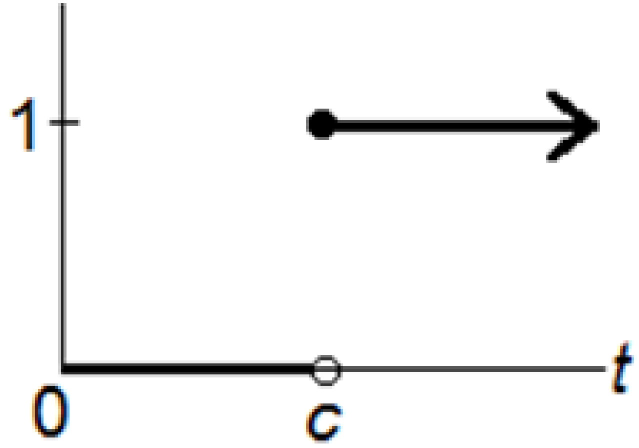
Laplace Transforms of Discontinuous Forcing Functions

MAT 275

We need a better way to describe functions with discontinuities. We use the **Heaviside Function**, which is

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

The graph looks like this:

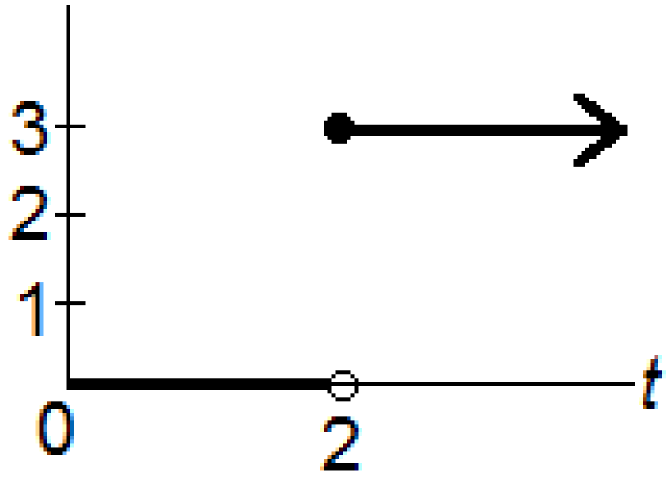


It's "off" (= 0) when $t < c$, then is "on" (= 1) when $t \geq c$.

Using coefficients, we can control the size of the "jumps". For example, $y = 3u_2(t)$ is equivalent to

$$y = \begin{cases} 0, & t < 2 \\ 3, & t \geq 2 \end{cases}$$

The jump discontinuity occurs at $t = 2$, and has a vertical change of 3 units:



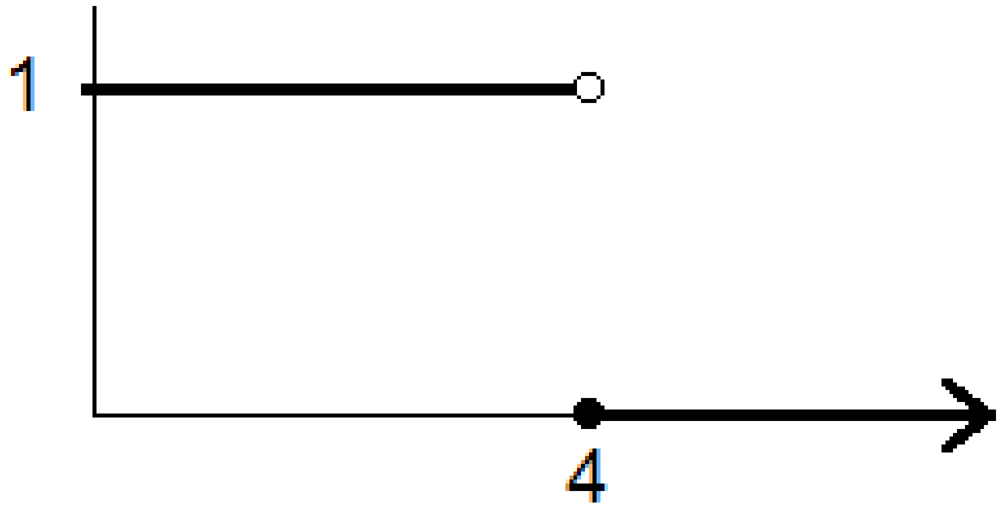
Note how much cleaner $3u_2(t)$ is in expressing the piecewise model.

Example: Sketch $y = 1 - u_4(t)$.

Solution:

- When $0 \le t < 4$, then $u_4(t) = 0$, and $y = 1 - 0 = 1$.
- When $t \ge 4$, then $u_4(t) = 1$, and $y = 1 - 1 = 0$.

The graph looks like this:



Example: Sketch $y = 2 - 3u_1(t) + 5u_3(t) - u_5(t)$.

Solution: There are three jumps, at $t = 1, 3$ and 5.

- For $0 \leq t < 1$, all three of the u terms are 0. Thus, $y = 2 - 3(0) + 5(0) - (0) = 2$.

- For $1 \leq t < 3$, we have $u_1(t) = 1$ but the other u terms are 0.

  Thus, $y = 2 - 3(1) + 5(0) - (0) = -1$.

- For $3 \leq t < 5$, we have $u_3(t) = 1$. Note that $u_1(t) = 1$ (it stays "on") but that $u_5(t) = 0$. Thus, $y = 2 - 3(1) + 5(1) - (0) = 2 - 3 + 5 = 4$.

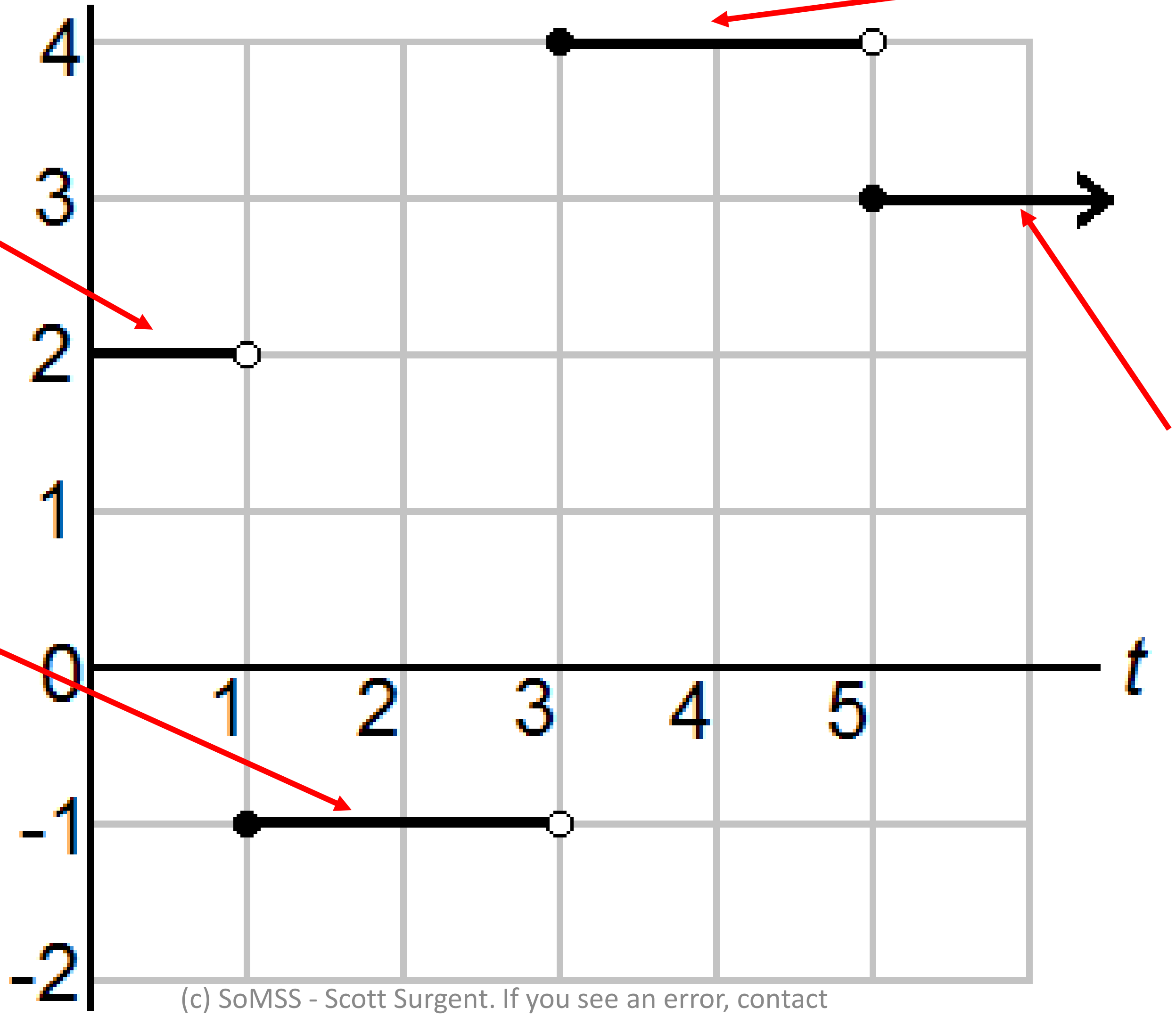
- For $t \geq 5$, we have $u_5(t) = 1$. Note that $u_1(t) = 1$ and that $u_3(t) = 1$. All of the u terms are now "on". Thus, $y = 2 - 3(1) + 5(1) - (1) = 2 - 3 + 5 - 1 = 3$.

The graph is on the next slide.

Sketch $y = 2 - 3u_1(t) + 5u_3(t) - u_5(t)$.

When $0 \le t < 1$, then $y = 2$.

When $1 \le t < 3$, then $y = -1$. Note a "jump" of –3 units from 2 to –1.



When $3 \le t < 5$, then $y = 4$. Note a "jump" of 5 units from –1 to 4.

When $t \ge 5$, then $y = 3$. Note a "jump" of –1 units from 4 to 3.

The closed point always occurs at the left endpoint of the interval!

Combining $u_c(t)$ notation with non-constant functions.

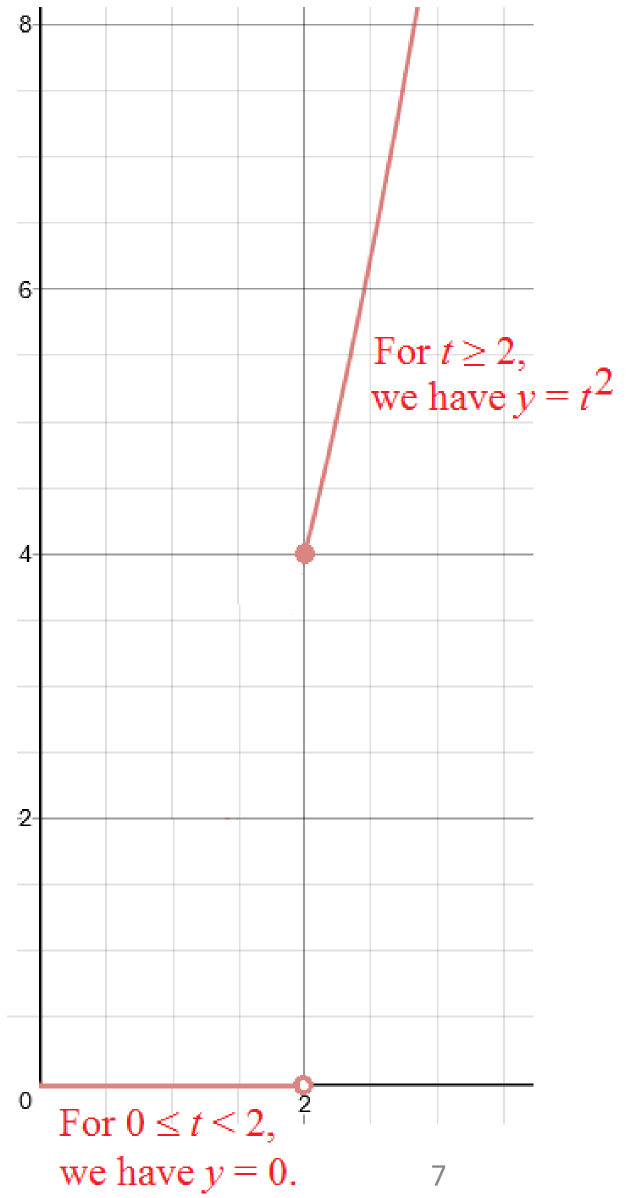
Example: Sketch $y = u_2(t)t^2$.

Solution: When $0 \leq t < 2$, we have $u_2(t) = 0$, so that $y = 0t^2 = 0$.

When $t \geq 2$, we have $u_2(t) = 1$, so that $y = 1t^2 = t^2$.

This is equivalent to the piecewise notation $y = \begin{cases} 0, & t < 2 \\ t^2, & t \geq 2 \end{cases}$.

The graph is 0 when $0 \leq t < 2$, and then at $t = 2$, the parabola t^2 "starts" here and continues upward.



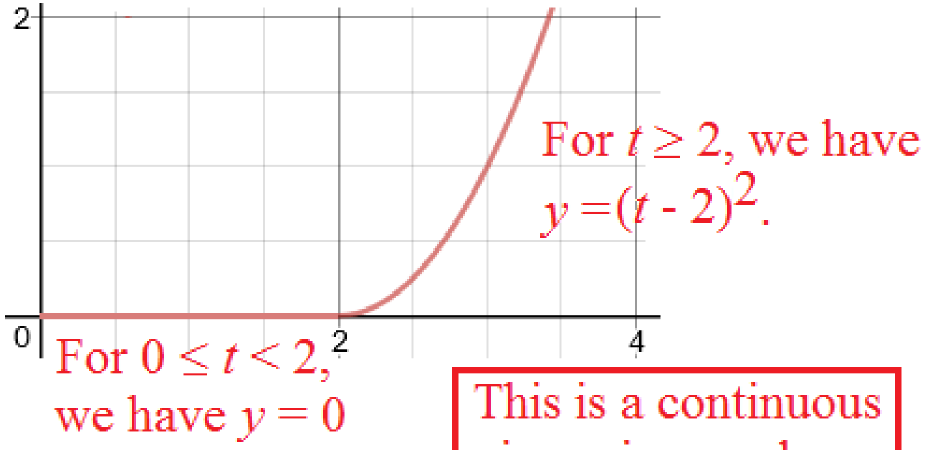
The $u_c(t)$ notation is useful for combining two or more function types into a single function, where continuity is desired.

Example: Sketch $y = u_2(t)(t-2)^2$.

Solution. When $0 \le t < 2$, we have $u_2(t) = 0$, so that $y = 0(t-2)^2 = 0$.

When $t \ge 2$, we have $u_2(t) = 1$, so that $y = 1(t-2)^2 = (t-2)^2$.

Here, the graph $(t-2)^2$ is a shift of t^2 two units to the right. It "starts" at (2,0). Note that is it continuous with the portion $y = 0$, where $0 \le t < 2$.



This is a continuous piecewise graph.

When taking the Laplace Transform of these types of functions, we need to "build in" the shift first!

Laplace Transform of y = $u_c(t)f(t-c)$

Start with: $L\{u_c(t)f(t-c)\} = \int_0^\infty u_c(t)f(t-c)e^{-st}\,dt\,.$

Since $u_c(t) = 0$ when $0 \le t < c$, and $u_c(t) = 1$ when $t \ge c$, we have

$$L\{u_c(t)f(t-c)\} = \int_c^\infty f(t-c)e^{-st}\,dt\,.$$

The integral from 0 to c is just 0

Now, let $w = t - c$, so that $t = w + c$. This is essentially a shift of c units to the left. Furthermore, $dw = d(t-c) = dt - dc = dt$, since $dc = 0$. We now have

$$\int_c^\infty f(t-c)e^{-st}\,dt = \int_0^\infty f(w)e^{-s(w+c)}dw\,.$$

Note that $e^{-s(w+c)} = e^{-sw-sc} = e^{-sw}e^{-sc}$, and that e^{-sc} is constant (it has no w).

Thus, we have

$$\int_0^\infty f(w)e^{-s(w+c)}dw = e^{-sc}\int_0^\infty f(w)e^{-sw}dw\,.$$

The integral on the right is just the Laplace Transform of $f(w)$:

$$L\{f(w)\} = \int_0^\infty f(w)e^{-sw}dw\,.$$

Here, w is just a dummy variable. It can be replaced with t. Thus, we have

$$L\{u_c(t)f(t-c)\} = e^{-sc}L\{f(t)\}.$$

For this to work, you **must** shift the function f by c units first, then evaluate the Laplace Transform as though the c was not present.

Example: Find $L\{u_3(t)\}$.

Solution: We can treat $u_3(t)$ as $u_3(t) = u_3(t) \cdot 1$. Shifting $y = 1$ left or right makes no difference, so we can proceed:

$$L\{u_3(t)\} = e^{-3s}L\{1\} = e^{-3s}\left(\frac{1}{s}\right) = \frac{e^{-3s}}{s}\ .$$

Example: Find $L\{u_5(t)(t-5)^3\}$.

Solution: Here, $c = 5$, so the function t^3 must be shifted 5 units to the right, and we see that $(t-5)^3$ already has the shift "built in". Thus, we have

$$L\{u_5(t)(t-5)^3\} = e^{-5s}L\{t^3\} = e^{-5s}\left(\frac{3!}{s^{3+1}}\right) = \frac{6e^{-5s}}{s^4}\ .$$

Example: Find $L\{u_3(t)t^2\}$.

Solution: Here, we have $c = 3$, so we need to rewrite t^2 so that is has a shift of 3 units to the right. We do this by writing $t^2 = (t - 3 + 3)^2$.

Now, multiply (FOIL) by grouping $t - 3$ and 3 separately:

$$t^2 = (t - 3 + 3)^2 = (t - 3)^2 + 2(t - 3)(3) + 3^2$$

Therefore, with a shift of 3 units to the right built in, $t^2 = (t - 3)^2 + 6(t - 3) + 9$.

Thus,

$$L\{u_3(t)t^2\} = e^{-3s}L\{t^2 + 6t + 9\} = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).$$

Remember, once you build in the shift, take the Laplace Transform of each term as though the shift was not there.

Example: Find $L\{u_{\pi/4}(t)\sin(t)\}$.

Solution: Since $c = \frac{\pi}{4}$, we build in the shift by writing $\sin(t)$ as $\sin\left(t - \frac{\pi}{4} + \frac{\pi}{4}\right)$.

Now, we use the identity $\sin(a+b) = \sin a \cos b + \cos a \sin b$, where $a = t - \frac{\pi}{4}$ and $b = \frac{\pi}{4}$:

$$\sin\left(t - \frac{\pi}{4} + \frac{\pi}{4}\right) = \sin\left(t - \frac{\pi}{4}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(t - \frac{\pi}{4}\right)\sin\left(\frac{\pi}{4}\right).$$

Recall that $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and that $\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. So now we have

$$\sin\left(t - \frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\sin\left(t - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\cos\left(t - \frac{\pi}{4}\right).$$

Now the shifts are built in. We can now find the Transform (next slide).

Solution, continued.

This is sin(*t*) after the shift has been built in, from the last slide.

$$L\{u_{\pi/4}(t)\sin(t)\} = e^{-(\pi/4)s} L\left\{\frac{\sqrt{2}}{2}\sin\left(t-\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2}\cos\left(t-\frac{\pi}{4}\right)\right\}$$

Factor any constants to the front.

$$= \frac{\sqrt{2}}{2} e^{-(\pi/4)s} L\{\sin(t)+\cos(t)\}$$

It is now "safe" to ignore the shift.

$$= \frac{\sqrt{2}}{2} e^{-(\pi/4)s}\left[\left(\frac{1}{s^2+1}\right) + \left(\frac{s}{s^2+1}\right)\right]$$

There are the Laplace Transforms of sin(*t*) and cos(*t*), respectively.

$$= \frac{\sqrt{2}}{2} e^{-(\pi/4)s}\left(\frac{s+1}{s^2+1}\right).$$

Simplify with a common denominator

Inverting Laplace Transforms with $u_c(t)$ notation.

Hint: When you see e^{-cs} in the Laplace Transform, this means there is a $u_c(t)$ in the resulting inversion (function).

Example: Find $L^{-1}\left\{\frac{5e^{-6s}}{s^2}\right\}$.

Solution: The 5 can be moved out front: $L^{-1}\left\{\frac{5e^{-6s}}{s^2}\right\} = 5L^{-1}\left\{\frac{e^{-6s}}{s^2}\right\}$.

The e^{-6s} suggests that there is a $u_6(t)$ in the resulting function.

Thus, we need to invert $\frac{1}{s^2}$. Note that $L\{t\} = \frac{1}{s^2}$, so that $L^{-1}\left\{\frac{1}{s^2}\right\} = t$.

The shift needs to be put back in. Thus, we have $L^{-1}\left\{\frac{5e^{-6s}}{s^2}\right\} = 5u_6(t)(t-6)$.

Example: Find $L^{-1}\left\{\frac{e^{-2s}}{s^4}\right\}$.

Solution: The e^{-2s} suggests that $u_2(t)$ is in the function. The shift is c = 2 units.

Ignore the e factor for the moment, we find $L^{-1}\left\{\frac{1}{s^4}\right\}$. Recall the general formula $L\{t^n\} = \frac{n!}{s^{n+1}}$.

This suggests that n = 3. so we need 3! = 6 in the numerator and its reciprocal outside:

$$L^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{6}L^{-1}\left\{\frac{6}{s^4}\right\} = \frac{1}{6}t^3.$$

To perform the inversion, we need the shift of 2 units in with the variable t. Thus,

$$L^{-1}\left\{\frac{e^{-2s}}{s^4}\right\} = \frac{1}{6}u_2(t)(t-2)^3.$$

Example: Find $L^{-1}\left\{\frac{e^{-s}}{s^2+9}\right\}$.

Solution: The e^{-s} suggests that $u_1(t)$ is in the result and that $c = 1$ is the shift.

Recall that $L\{\sin(3t)\} = \frac{3}{s^2+9}$. Thus, $L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$.

Note that we multiplied by a 3 on the "inside" and a 1/3 on the "outside" so that the inversion could be performed.

The shift $c = 1$ is then accounted for in the sine function. Thus,

$$L^{-1}\left\{\frac{e^{-s}}{s^2+9}\right\} = \frac{1}{3}u_1(t)\sin\big(3(t-1)\big).$$