Systems of Ordinary Differential Equations

Case III: Real eigenvalues of repeated multiplicity

MAT 275

Solve $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} \mathbf{x}$.

Find the eigenvalues:

$$\det \begin{bmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(5-\lambda) + 1 = 0$$

$$\lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

Thus, $\lambda = 4$ is a repeated (multiplicity 2) eigenvalue.

The eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

One term of the solution is $\mathbf{x}^{(1)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$.

The second solution is found by "guessing" that $\mathbf{x}^{(2)} = \mathbf{a}te^{4t} + \mathbf{b}e^{4t}$, where $\mathbf{a}$ and $\mathbf{b}$ are two vectors to be determined.

(Note: if you simply guess $\mathbf{x} = \mathbf{a}te^{4t}$, it won't work. Try it.)

Thus, $\left(\mathbf{x}^{(2)}\right)' = 4\mathbf{a}te^{4t} + \mathbf{a}e^{4t} + 4\mathbf{b}e^{4t}$, and we have

$$4\mathbf{a}te^{4t} + \mathbf{a}e^{4t} + 4\mathbf{b}e^{4t} = A(\mathbf{a}te^{4t} + \mathbf{b}e^{4t})$$

where A is the matrix.

We now equate both sides according to the terms te^{4t} and e^{4t}:
$$\begin{aligned} 4\mathbf{a}te^{4t} &= A\mathbf{a}te^{4t} \\ (\mathbf{a} + 4\mathbf{b})e^{4t} &= A\mathbf{b}e^{4t} \end{aligned}$$

The first line means that $4\mathbf{a} = A\mathbf{a}$.

However, recall that 4 is an eigenvalue of A, and if $4\mathbf{a} = A\mathbf{a}$, then $\mathbf{a}$ must be the eigenvector $\mathbf{v}_1$.

In the second line, we have $\mathbf{a} + 4\mathbf{b} = A\mathbf{b}$.

Replace $\mathbf{a}$ with $\mathbf{v}_1$ and re-arrange, we have $A\mathbf{b} - 4\mathbf{b} = \mathbf{v}_1$.

The left side is similar to solving for an eigenvector, but instead of setting the right side to $\mathbf{0}$, we set it to $\mathbf{v}_1$.

Thus, $A\mathbf{b} - 4\mathbf{b} = \mathbf{v}_1$ is the same as $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{b} = \mathbf{v}_1$.

Letting $\mathbf{b} = \begin{bmatrix} x \\ y \end{bmatrix}$ for the moment, we have

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \text{or} \quad -x - y = 1.$$

Here, we must solve generically for all possible **b**.

If we let $x = k$, then $y = -1 - k$, so therefore,

$$\mathbf{b} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k \\ -1 - k \end{bmatrix} = \begin{bmatrix} 0 + 1k \\ -1 - 1k \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} k.$$

Note that the vector in front of the k is $\mathbf{v}_1$ again (or a multiple thereof). This is a check of your work.

Thus, another solution is

$$\mathbf{x}^{(2)} = \mathbf{a}te^{4t} + \mathbf{b}e^{4t} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{4t} + \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} k \right) e^{4t}.$$

The general solution is

$$\mathbf{x} = \mathbf{x}^{(1)} + \mathbf{x}^{(2)} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t} + c_2 \left[\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{4t} + \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} k \right) e^{4t} \right].$$

But wait, there's more!

Note that the term $\begin{bmatrix} 1 \\ -1 \end{bmatrix} ke^{4t}$ is just a multiple of $c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$.

We can either "ignore" it, or combine them together into a single term.
Either way, it's no longer written.

The final general solution is $\mathbf{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t} + c_2 \left[\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{4t} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} e^{4t} \right].$

To summarize:

In the differential equation of the form $\mathbf{x}' = A\mathbf{x}$, where A is a 2 by 2 matrix with a real eigenvalue λ of multiplicity 2 and an eigenvector $\mathbf{v}_1$, the general solution is

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda t} + c_2\left[\mathbf{v}_1 t e^{\lambda t} + \mathbf{b} e^{\lambda t}\right],$$

Where $\mathbf{b}$ is found by solving $A\mathbf{b} - \lambda\mathbf{b} = \mathbf{v}_1$, being sure to find $\mathbf{b}$ generically in terms of a parameter k.