Step Functions, Shifting and Laplace Transforms

The basic step function (called the *Heaviside Function*) is

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}.$$

It is "off" (0) when $t < c$, the "on" (1) when $t \geq c$. Don't let the notation confuse you. The function $u_c(t)$ is either 0 and 1, nothing more.

If $f(t)$ is a function, then we can shift it so that it "starts" at $t = c$. This results in the function

$$y = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}.$$

Using the step notation, this same function (now called $g(t)$) can be written

$$g(t) = u_c(t)f(t-c).$$

The Laplace Transform is

$$L\{g(t)\} = L\{u_c(t)f(t-c)\} = e^{-cs}L\{f(t)\}.$$

Be sure the shift is already accounted for beforehand, then take the transform of the function as normally done.

Example: Find the Laplace transform of $y = u_2(t)(t-2)^2$.

Solution: Here, $y = 0$ for $t < 2$, then $u_2(t) = 1$ for $y \geq 2$. Thus, it turns "on" the function $(t-2)^2$, which is the graph of $f(t) = t^2$ shifted 2 units to the right. This is in the correct form to use the Laplace transform. The Laplace Transform of t^2 is $L\{t^2\} = \frac{2}{s^3}$. Therefore, the Laplace transform of $g(t) = u_2(t)(t-2)^2$ is

$$L\{g(t)\} = L\{u_2(t)(t-2)^2\} = e^{-2s}L\{t^2\} = e^{-2s}\left(\frac{2}{s^3}\right) = \frac{2e^{-2s}}{s^3}.$$

Example: Find the Laplace Transform of $y = u_3(t)t^2$.

Solution: Note that this is *not* the function t^2 shifted right 3 units. This is actually the function t^2 still centered at $t = 0$, staying "off" until $t = 3$, then it turns on. In other words, t^2 has not been shifted right 3 units like the form requires.

We work in the shift as follows:

$$g(t) = u_3(t)(t-3+3)^2$$

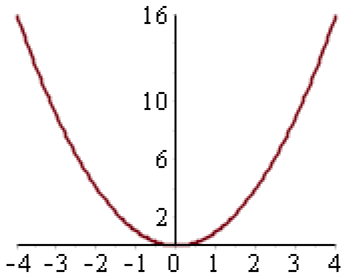
We expand $(t-3+3)^2$ by grouping $t-3$ as one term, 3 as the other term:

$$(t-3+3)^2 = (t-3)^2 + 6(t-3) + 9.$$

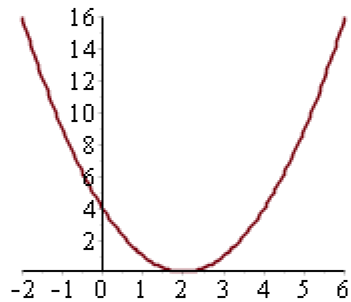
So the function becomes $g(t) = u_3(t)t^2 = u_3(t)(t-3+3)^2 = u_3(t)[(t-3)^2 + 6(t-3) + 9]$.

Now we can use the Laplace Transform formulas:

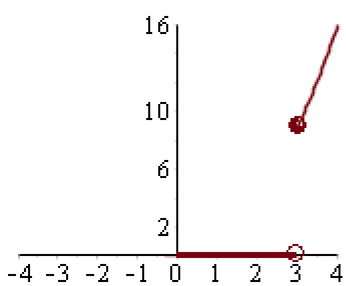
$$L\{u_3(t)[(t-3)^2 + 6(t-3) + 9]\} = e^{-3s}[L\{t^2\} + 6L\{t\} + 9] = e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right).$$



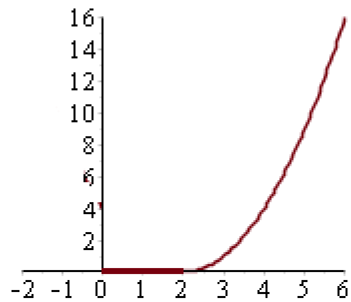
$f(t) = t^2$



$y = (t-2)^2$



$y = u_3(t)t^2$

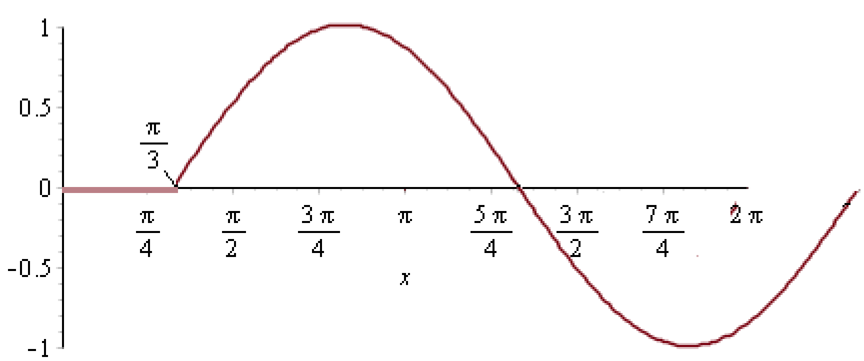


$y = u_2(t)(t-2)^2$

Note the differences in the graphs of $y = t^2$ and $y = (t-2)^2$ (above left and right), then the graphs of $y = u_3(t)t^2$ and $y = u_2(t)(t-2)^2$ (below left and right), where the $u_c(t)$ is the "on-off" switch, and the graph of t^2 has not been shifted (left) and has been shifted (right).

Example: Find $L\left\{u_{\pi/3}(t)\sin\left(t - \frac{\pi}{3}\right)\right\}$.

Solution: The graph of $y = u_{\pi/3}(t)\sin\left(t - \frac{\pi}{3}\right)$ is the sine function shifted to the right $\frac{\pi}{3}$ units, as shown below:

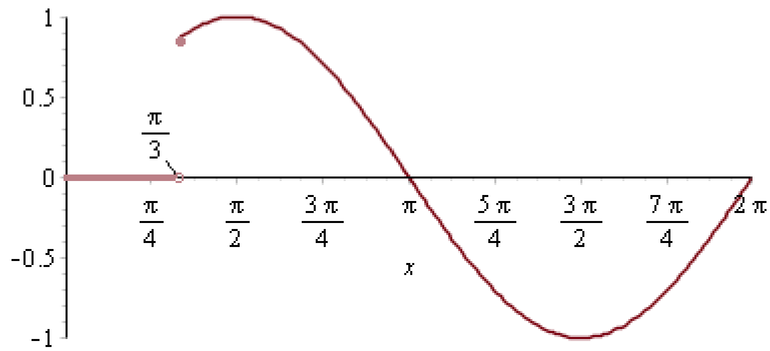


The shift is accounted for, so

$$L\left\{u_{\pi/3}(t)\sin\left(t-\frac{\pi}{3}\right)\right\} = e^{-(\pi/3)s}L\{\sin t\}$$
$$= e^{-(\pi/3)s}\left(\frac{1}{s^2+1}\right)$$
$$= \frac{e^{-(\pi/3)s}}{s^2+1}.$$

Example: Find $L\{u_{\pi/3}(t)\sin t\}$.

Solution: The graph of $y = u_{\pi/3}(t)\sin t$ is shown below. Note that it is the sine function, but *not* shifted. It simply is "off" until $t = \frac{\pi}{3}$, at which time it turns "on" and follows the graph of $\sin t$ as normal.



The shift $t-\frac{\pi}{3}$ is not present in the $\sin t$ as is needed to perform this transform. We put it in as follows:

$$\sin t = \sin\left(t-\frac{\pi}{3}+\frac{\pi}{3}\right).$$

We then use the sine-sum identity, $\sin(a+b) = \sin a \cos b + \cos a \sin b$:

$$\sin\left[\left(t-\frac{\pi}{3}\right)+\frac{\pi}{3}\right] = \sin\left(t-\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(t-\frac{\pi}{3}\right)\sin\frac{\pi}{3}.$$

Recall from trigonometry that $\cos\frac{\pi}{3} = \frac{1}{2}$ and that $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$. Therefore,

$$\begin{aligned}\sin\left[\left(t-\frac{\pi}{3}\right)+\frac{\pi}{3}\right] &= \sin\left(t-\frac{\pi}{3}\right)\cos\frac{\pi}{3} + \cos\left(t-\frac{\pi}{3}\right)\sin\frac{\pi}{3} \\ &= \frac{1}{2}\sin\left(t-\frac{\pi}{3}\right) + \frac{\sqrt{3}}{2}\cos\left(t-\frac{\pi}{3}\right).\end{aligned}$$

Therefore, to find $L\{u_{\pi/3}(t)\sin t\}$, we find $L\left\{u_{\pi/3}(t)\left[\frac{1}{2}\sin\left(t-\frac{\pi}{3}\right)+\frac{\sqrt{3}}{2}\cos\left(t-\frac{\pi}{3}\right)\right]\right\}$.

Recall that $L\{\sin t\} = \frac{1}{s^2+1}$ and that $L\{\cos t\} = \frac{s}{s^2+1}$. Since the shifts are now accounted for, we have:

$$\begin{aligned}L\{u_{\pi/3}(t)\sin t\} &= L\left\{u_{\pi/3}(t)\left[\frac{1}{2}\sin\left(t-\frac{\pi}{3}\right)+\frac{\sqrt{3}}{2}\cos\left(t-\frac{\pi}{3}\right)\right]\right\} \\ &= e^{-(\pi/3)s}\left[\frac{1}{2}L\{\sin t\}+\frac{\sqrt{3}}{2}L\{\cos t\}\right] \\ &= e^{-(\pi/3)s}\left[\frac{1}{2}\left(\frac{1}{s^2+1}\right)+\frac{\sqrt{3}}{2}\left(\frac{s}{s^2+1}\right)\right] \\ &= \frac{1}{2}e^{-(\pi/3)s}\left(\frac{1+\sqrt{3}s}{s^2+1}\right).\end{aligned}$$