Spring-Mass Systems

MAT 275

An object has weight, where weight is mass times gravity: $w = mg$. Here, g is 32 ft/s^2.

Hooke's Law: the force F_h to extend a spring a distance L feet is proportional to L, so that $F_h = kL$, where k is a constant of proportionality.

If the object is attached at the end of a spring, and is allowed to rest (not bob up and down), then the two forces cancel: the spring wants to contract in a direction opposite the weight, so that

$$mg - kL = 0, \qquad \text{which gives} \qquad mg = kL.$$

Suppose the object is pulled down and let go. It then starts to bob up and down. Let $u(t)$ be the distance (in feet) at t seconds of the object from rest, where down is positive.

The object extended the spring L feet, then the spring's length is also affected by the object's motion, so that $F_h = k\big(L + u(t)\big)$.

The second derivative of displacement is $u''(t)$, which describes the object's acceleration at time t. Using the familiar formula $F = ma$, we now have $F = mu''(t)$.

There is a resistance force, F_d, defined as a force (in lbs) acting against the bobbing mass when the mass is at some speed, given by $u'(t)$. The force is proportional to the speed but in the opposite direction, so that $F_d = -\gamma u'(t)$.

The sum of all forces must equal $mu''(t)$:

$$mu''(t) = F_d + F_h$$
$$mu''(t) = mg - k\big(L + u(t)\big) - \gamma u'(t)$$
$$mu''(t) = mg - kL - ku(t) - \gamma u'(t)$$
$$mu''(t) = -ku(t) - \gamma u'(t)$$

$mg - kL = 0$

Thus, we have $mu''(t) + \gamma u'(t) + ku(t) = 0$ (collecting terms to one side).

From the last screen, we have $mu''(t) + \gamma u'(t) + ku(t) = 0$.

Remember, $w = mg$ so that $m = \frac{w}{g}$, and $k = \frac{\text{weight}}{\text{initial displacement}}$.

Written out fully, the form of the equation is

$$\left(\frac{\text{weight}}{\text{gravity}}\right)u''(t) + \left(\frac{\text{resistance}}{\text{velocity}}\right)u'(t) + \left(\frac{\text{weight}}{\text{init. displacement}}\right)u(t) = 0.$$

The units are:

$$\left(\frac{\text{lbs}}{\text{ft/s}^2}\right)(\text{ft/s}^2) + \left(\frac{\text{lbs}}{\text{ft/s}}\right)(\text{ft/s}) + \left(\frac{\text{lbs}}{\text{ft}}\right)(\text{ft}).$$

Everything simplifies into pounds (lbs), which is a force.

Example: A mass weighing 4 lbs stretches a spring 2 inches (1/6 feet). The mass is pulled down 6 more inches (1/2 foot) then released. When the mass is moving at 3 feet/second, the surrounding medium applies a resistance force of 6 lbs. Find the initial value problem that governs the motion of the bobbing mass, and solve for $u(t)$.

Solution: Use the form $\left(\frac{\text{weight}}{\text{gravity}}\right)u''(t)+\left(\frac{\text{resistance}}{\text{velocity}}\right)u'(t)+\left(\frac{\text{weight}}{\text{init.displacement}}\right)u(t)=0.$

Here, we have $w = 4$, $g = 32$, resistance = 6 when $v = 3$, and init. displacement 1/6:

$$\left(\frac{4}{32}\right)u''(t)+\left(\frac{6}{3}\right)u'(t)+\left(\frac{4}{1/6}\right)u(t)=0.$$

Simplified, we have $\frac{1}{8}u''(t)+2u'(t)+24u(t)=0$. Multiplying through by 8, we have

"released" means no initial velocity.

$$u''(t)+16u'(t)+192u(t)=0, \qquad \text{where} \qquad u(0)=\frac{1}{2}, \qquad u'(0)=0.$$

The auxiliary polynomial is $r^2 + 16r + 192 = 0$. Using the quadratic formula, we have

$$r = \frac{-16 \pm \sqrt{16^2 - 4(1)(192)}}{2(1)} = \frac{-16 \pm \sqrt{-512}}{2} = \frac{-16 \pm 16i\sqrt{2}}{2} = -8 \pm 8i\sqrt{2}.$$

The general solution is

$$u(t) = e^{-8t}\left(C_1 \cos\left(8\sqrt{2}t\right) + C_2 \sin\left(8\sqrt{2}t\right)\right).$$

When $u(0) = \frac{1}{2}$, the sine term vanishes, so $C_1 = \frac{1}{2}$.

We now have

$$u(t) = e^{-8t}\left(\frac{1}{2}\cos\left(8\sqrt{2}t\right) + C_2 \sin\left(8\sqrt{2}t\right)\right).$$

From the last slide, we have $u(t) = e^{-8t}\left(\frac{1}{2}\cos\left(8\sqrt{2}t\right) + C_2\sin\left(8\sqrt{2}t\right)\right)$. To find C_2, we take its derivative and use the other initial condition, $u(0) = 0$. The derivative is

$$u'(t) = e^{-8t}\left(-4\sqrt{2}\sin\left(8\sqrt{2}t\right) + C_2 8\sqrt{2}\cos\left(8\sqrt{2}t\right)\right) - 8e^{-8t}\left(\frac{1}{2}\cos\left(8\sqrt{2}t\right) + C_2\sin\left(8\sqrt{2}t\right)\right).$$

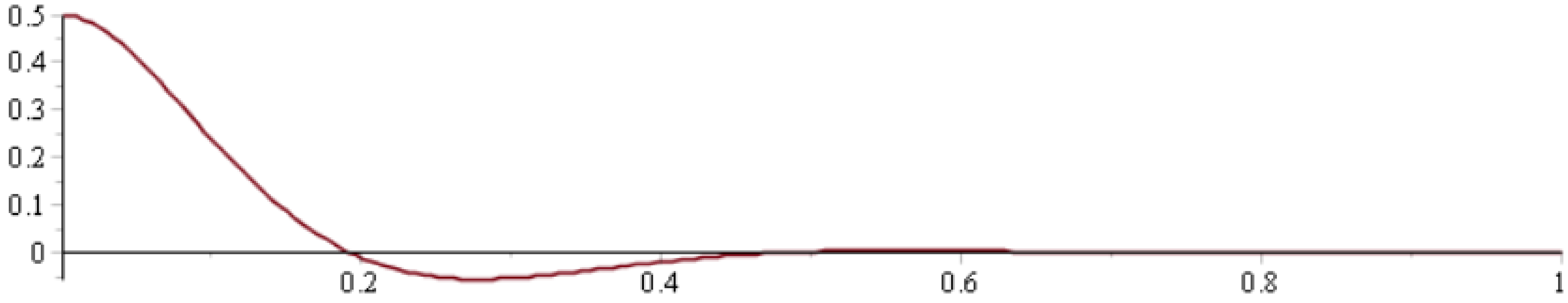
But when $t = 0$, nice things happen. We have:

$$0 = C_2 8\sqrt{2} - 4, \qquad \text{which gives} \qquad C_2 = \frac{\sqrt{2}}{4}.$$

Thus, the equation that models the bobbing spring is

$$u(t) = e^{-8t}\left(\frac{1}{2}\cos\left(8\sqrt{2}t\right) + \frac{\sqrt{2}}{4}\sin\left(8\sqrt{2}t\right)\right)$$

Here is a graph of the bobbing spring over a 1-second interval of time:



The spring bobs a couple times but quickly comes back to the rest state due to the viscous "dampening" force.

Because the mass was able to bob back to the rest state and beyond (i.e. "up and down"), but the amplitude trends to 0 as t increases, this is called an **damped** system.

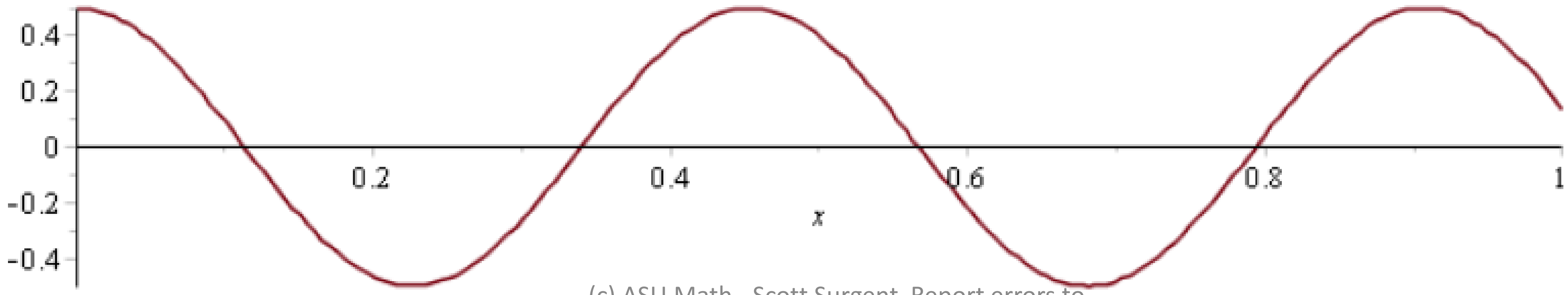
Note that the e^{-8t} factor in the solution acts as an "envelope", governing the amplitude. As t increases, e^{-8t} decreases to 0.

Example: Same mass and spring as before, but there is no resistive force. That is, $\gamma = 0$. Find $u(t)$ and sketch its graph.

Solution: The differential equation is $\frac{1}{8}u''(t) + 24u(t) = 0$, the same initial conditions.

This simplifies to $u''(t) + 192u(t) = 0$. The auxiliary polynomial is $r^2 + 192 = 0$, which has roots $r = \pm 8i\sqrt{3}$. The general solution is $u(t) = C_1 \cos\left(8\sqrt{3}t\right) + C_2 \sin\left(8\sqrt{3}t\right)$.

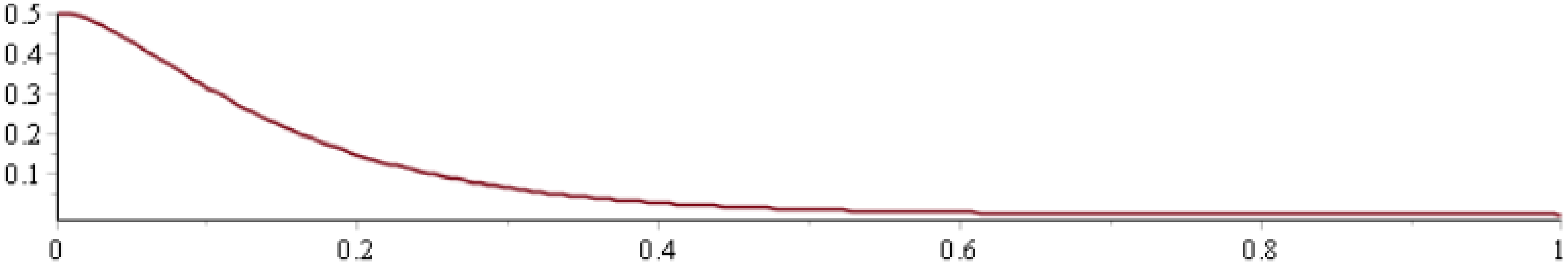
Once the initial conditions are worked in, the particular solution is $u(t) = \frac{1}{2}\cos\left(8\sqrt{3}t\right)$. Below is a 1-second graph of the bobbing mass. It never loses amplitude, bobbing forever.



Example: Suppose the resistive force is twice the original (12 lbs instead of 6 lbs). Find $u(t)$.

Solution: The differential equation is $u''(t) + 32u'(t) + 192u(t) = 0$. The auxiliary polynomial is $r^2 + 32r + 192 = 0$, which factors as $(r + 24)(r + 8) = 0$, with solutions $r = -24$ and $r = -8$.

The general solution is $u(t) = C_1 e^{-8t} + C_2 e^{-24t}$. When the initial conditions are figured in, the particular solution is $u(t) = \frac{3}{4}e^{-8t} - \frac{1}{4}e^{-24t}$. One second of motion is shown below:



The surrounding viscous medium is so dense the mass never bobs. It just slowly moves back to rest state. This is an **overdamped** system.

There are three possible outcomes:

- The auxiliary polynomial has two complex conjugate solutions of the form $r = a \pm bi$. If $a = 0$, then the system is **undamped** and the mass bobs up and down forever. If $a < 0$, then the e^{at} factor of the solution acts as an “envelope” function, approaching 0 as t increases, and thus **damping** the bobbing nature of the mass. (Note that a is never positive in these problems since that would result in an envelope function that grows with time. The amplitude would increase, not decrease.)
- The auxiliary polynomial has two real but different solutions. This is an **overdamped** system. The mass never actually bobs. It just slowly moves back to the rest state asymptotically.
- The auxiliary polynomial has one real and repeated root. This is called a **critically-damped** system.

To determine when a system is critically damped, we solve the generic auxiliary polynomial form $mr^2 + \gamma r + k = 0$ using the quadratic formula, getting

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}.$$

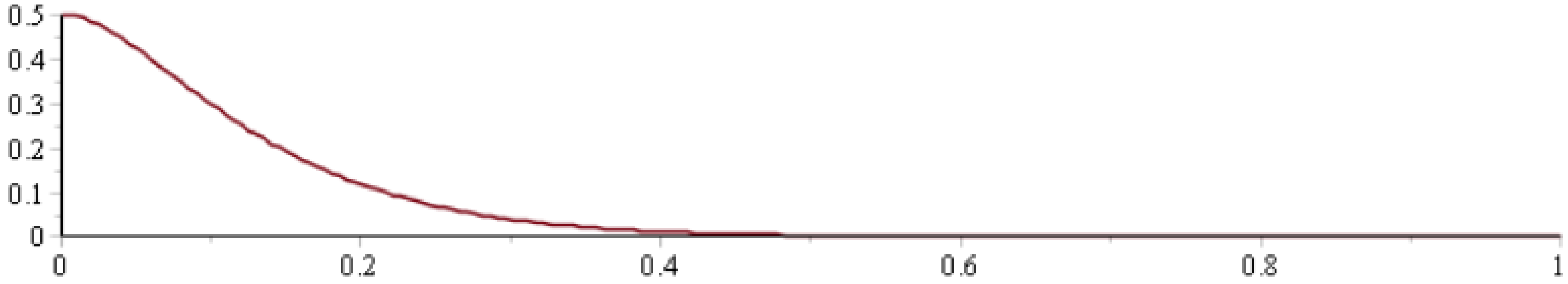
We then set the discriminant to 0: $\gamma^2 - 4mk = 0$, which implies that $\gamma = 2\sqrt{mk}$. In the original example, we have m = 1/8 and k = 24, so that $\gamma = 2\sqrt{\left(\frac{1}{8}\right)(24)} = 2\sqrt{3}$ would result in a critically-damped system.

Recall that the original resistance was 6 lbs when velocity was 3 ft/s. If we set the resistance to $6\sqrt{3}$ lbs at a velocity of 3 ft/s, then we get $\gamma = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$, which will cause the auxiliary polynomial to have just one root. (next slide)

We get the differential equation $\frac{1}{8}u'' + 2\sqrt{3}u' + 24u = 0$ which has solutions $r = -8\sqrt{3}$, multiplicity 2. The particular solution (with the same initial conditions as before) is

$$u(t) = e^{-8\sqrt{3}t}\left(4\sqrt{3}t + \frac{1}{2}\right).$$

One second of motion is:



It does not look any different from an overdamped system. This is the “boundary” of damped-overdamped. If the resistance force is less than $6\sqrt{3}$ lbs, then we have a bobbing mass.

An External Forcing Function. Suppose the original example has an external forcing function that imparts force into the system (e.g. by shaking it rhythmically). Suppose we have a forcing function $\sin\left(\frac{1}{2}t\right)$. Thus, the differential equation is

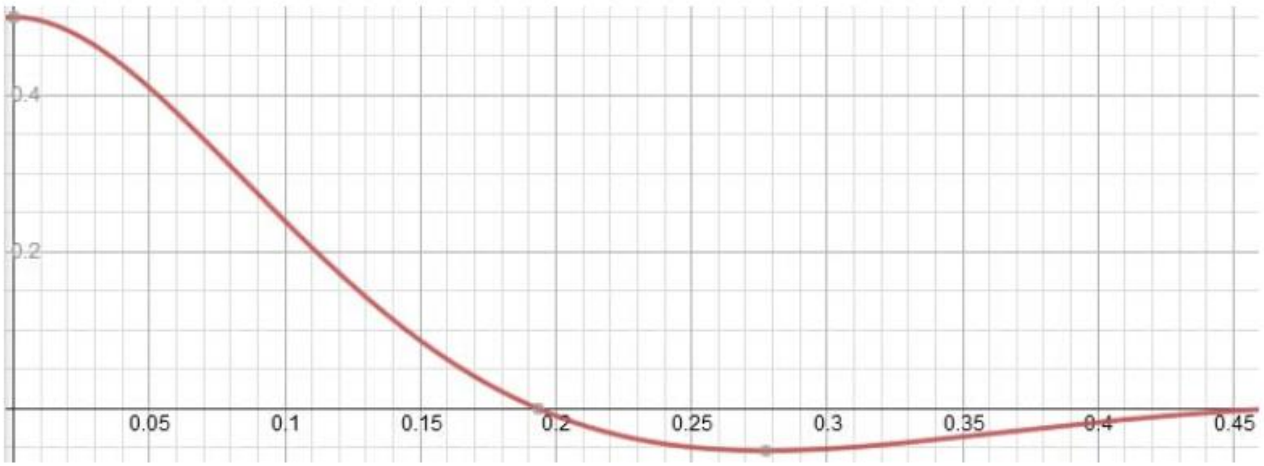
$$u''(t) + 16u'(t) + 192u(t) = \sin\left(\frac{1}{2}t\right), \qquad u(0) = \frac{1}{2}, \qquad u'(0) = 0.$$

The general solution is

$$u(t) = e^{-8t}\left(0.5002172\cos\left(8\sqrt{2}t\right) + 0.3535\sin\left(8\sqrt{2}t\right)\right) + 0.005206\sin\left(\frac{1}{2}t\right) = 0.0002172\cos\left(\frac{1}{2}t\right).$$

The graphs are on the next slide.

Here's the motion of the mass for the first 0.5 second:



Then for the first 25 seconds. The forcing function "overwhelms" the natural decay of the unforced case, and the mass bobs according to the forcing function: