Separable Differential Equations

MAT 275

A **separable** differential equation is one that can be written so that the independent variable terms (along with its differential) are collected to one side of the equal sign, and the dependent variable terms (and its differential) to the other.

Example: $y' = xy^2$ is separable. It is first written as $\frac{dy}{dx} = xy^2$, then "separated":

$$\frac{dy}{y^2} = x\,dx.$$

To solve a separable differential equation, integrate both sides and solve for y (if possible). For example, we had $y' = xy^2$. This is separated as $\frac{dy}{y^2} = x\,dx$.

$\int \frac{dy}{y^2} = \int x\,dx$ Integrate both sides.

$-\frac{1}{y} = \frac{1}{2}x^2 + C$ Don't forget the constant of integration.

$\frac{1}{y} = C - \frac{1}{2}x^2$ Negate. The C "absorbs" the negative.

$y = \frac{1}{C-\frac{1}{2}x^2} = \frac{2}{C-x^2}$ Solve for y. Note that $2C$ is written as C.

Example: find the general solution of $y' = x + xy$.

Write y' as $\frac{dy}{dx}$: $\quad \frac{dy}{dx} = x + xy$

Factor: $\quad \frac{dy}{dx} = x(1+y)$

Separate: $\quad \frac{dy}{1+y} = x\,dx \qquad (y \neq -1)$

Integrate: $\quad \int \frac{dy}{1+y} = \int x\,dx \quad \rightarrow \quad \ln|1+y| = \frac{1}{2}x^2 + C$

Isolate y: $\quad |1+y| = e^{0.5x^2+C} \quad \rightarrow \quad |1+y| = Ce^{0.5x^2}$

$$1 + y = \pm Ce^{0.5x^2} = Ce^{0.5x^2} \qquad (\pm C = C)$$

Thus, $y = Ce^{0.5x^2} - 1$ is the general solution of $y' = x + xy$.

- The constant of integration C is just a generic constant at this point. It absorbs all constants that come near it, so to speak. For example, $e^C = C$, $-C = C$, $2C = C$, $\frac{1}{C} = C$, and so on.
- The C can be determined with an initial condition. For example, suppose we have $y' = x + xy$ with $y(0) = 3$. The general solution is $y = C^{0.5x^2} - 1$. To find C, let $x = 0$ and $y = 3$:

$$3 = Ce^{0.5(0)^2} - 1 \quad \rightarrow \quad 3 = C - 1 \quad \rightarrow \quad C = 4.$$

Thus, the particular solution is $y = 4e^{0.5x^2} - 1$.

Example: the rate of change in the value of a stock is inversely-proportional to the square of the value of that stock. If the stock's value was \$20 at noon, and was \$23 at 3 p.m., what is the stock's value at 5 p.m.?

$$V'(t) = \frac{k}{[V(t)]^2} \quad \rightarrow \quad \frac{dV}{dt} = \frac{k}{V^2}$$

$$V^2\, dV = k\, dt \quad \rightarrow \quad \int V^2\, dV = \int k\, dt$$

$$\frac{1}{3}V^3 = kt + C$$

Example: the rate of change in the value of a stock is inversely-proportional to the square of the value of that stock. If the stock's value was \$20 at noon, and was \$23 at 3 p.m., what is the stock's value at 5 p.m.?

$$\frac{1}{3}V^3 = kt + C$$
$$V^3 = 3kt + C$$
$$V(t) = \sqrt[3]{3kt + C}.$$

To find C, use (0,20), where $t = 0$ is noon: $20 = \sqrt[3]{3k(0) + C} \rightarrow 20^3 = C$

Thus, we have $V(t) = \sqrt[3]{3kt + 8000}$.

To find k, use (3,23): $23 = \sqrt[3]{3k(3) + 8000} \rightarrow 23^3 = 9k + 8000$

$$23^3 - 8000 = 9k \quad \rightarrow \quad 4167 = 9k \quad \rightarrow \quad k = \frac{4167}{9} = 463.$$

The particular solution is $V(t) = \sqrt[3]{3(463)t + 8000}$ or $V(t) = \sqrt[3]{1389t + 8000}$.

At 5 p.m., the stock's value is $V(5) = \sqrt[3]{1389(5) + 8000} \approx \$24.63$.

Integration Factors

There is a process by which most first-order linear differential equations can be solved. This uses an **integration factor**, denoted $\mu(x)$ (Greek letter “mu”).

The differential equation must be in the form

$$y' + f(x)y = g(x).$$

To find $\mu(x)$, we perform the following process (next slide)

Starting with $y' + f(x)y = g(x)$, multiply both sides by $\mu(x)$:

$$\mu(x)y' + \mu(x)f(x)y = \mu(x)g(x)$$

The left side is a product-rule derivative of $(\mu(x)y)$:

$$(\mu(x)y)' = \mu(x)y' + \mu'(x)y.$$

Thus, we have $\mu(x)y' + \mu'(x)y = \mu(x)y' + \mu(x)f(x)y$.

This forces $\mu'(x) = \mu(x)f(x)$. (*next slide*)

Now we find $\mu(x)$. From the last slide, we had $\mu'(x) = \mu(x)f(x)$.

This is a separable differential equation… so separate:

$$\frac{d\mu}{\mu(x)} = f(x)dx.$$

Integrating both sides, we have

$$\int \frac{d\mu}{\mu(x)} = \int f(x)dx\,.$$

(*next slide*)

After integration, we have

$$\ln \mu(x) = \int f(x)\, dx + C.$$

Here, we only need one form of the antiderivative, so we let $C = 0$. Taking base-e on both sides, we now know $\mu(x)$:

$$\mu(x) = e^{\int f(x) dx}.$$

Good news… you don't need to do all those steps each time. Just remember that if you have a differential equation of the form $y' + f(x)y = g(x)$, then find $\mu(x) = e^{\int f(x)dx}$.

Let's take a few seconds to relax and look at kittens:

So let's go back to the start: we have a differential equation of the form $y' + f(x)y = g(x)$.

We have found $\mu(x)$ and multiplied it to both sides:

$$\mu(x)y' + \mu(x)f(x)y = \mu(x)g(x).$$

The left side is compressed as a product rule derivative:

$$(\mu(x)y)' = \mu(x)g(x).$$

Now integrate both sides. Here, we need the constant of integration:

$$\int (\mu(x)y)' = \int \mu(x)g(x)dx + C.$$

Since $\int (\mu(x)y)' = \mu(x)y$, we can now solve for y:

$$\boxed{y = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}.}$$

You may memorize this as a formula.

Example: Find the general solution of $y' + \frac{2}{x}y = x$.

(Note that $f(x) = \frac{2}{x}$ and $g(x) = x$ and that $x \neq 0$)

Solution: First, we find $\mu(x)$:

$$\mu(x) = e^{\int \left(\frac{2}{x}\right)dx} = e^{2\ln x} = e^{\ln x^2} = x^2.$$

Now, use the formula $y = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}$: (*next slide*)

Solve $y' + \frac{2}{x}y = x$. From previous slide, we know that $\mu(x) = x^2$.

Now we use the formula $y = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}$:

$$y = \frac{\int x^2 x\, dx + C}{x^2} = \frac{\int x^3 dx + C}{x^2} = \frac{\frac{1}{4}x^4 + C}{x^2} = \frac{1}{4}x^2 + Cx^{-2}.$$

Thus, $y = \frac{1}{4}x^2 + Cx^{-2}$ is the general solution of $y' + \frac{2}{x}y = x$.

On the next slide, we'll check to be sure.

Check that $y = \frac{1}{4}x^2 + Cx^{-2}$ is the general solution of $y' + \frac{2}{x}y = x$. First, differentiate y:

$$y' = \frac{1}{2}x - 2Cx^{-3}$$

Now insert y' and y into the differential equation and simplify:

$$\left(\frac{1}{2}x - 2Cx^{-3}\right) + \frac{2}{x}\left(\frac{1}{4}x^2 + Cx^{-2}\right) = x$$
$$\left(\frac{1}{2}x - 2Cx^{-3}\right) + \left(\frac{1}{2}x + 2Cx^{-3}\right) = x$$
$$\left(\frac{1}{2}x + \frac{1}{2}x\right) + (-2Cx^{-3} + 2Cx^{-3}) = x$$
$$x + 0 = x.$$

Example: Solve the IVP: $y' + 2xy = 3x, \;\; y(0) = 5.$

IVP stands for "initial value problem"

Solution: Find $\mu(x)$: $\mu(x) = e^{\int 2x\, dx} = e^{x^2}.$

Now find y using the formula $y = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}$:

$$y = \frac{\int \mu(x)g(x)dx + C}{\mu(x)} = \frac{\int e^{x^2}(3x)dx + C}{e^{x^2}} = \frac{\frac{3}{2}e^{x^2} + C}{e^{x^2}} = \frac{3}{2} + Ce^{-x^2}.$$

To find C, use $y(0) = 5$: $5 = \frac{3}{2} + Ce^{-(0)^2} \rightarrow C = \frac{7}{2}.$

Thus, the particular solution is $\boxed{y = \frac{3}{2} + \frac{7}{2}e^{-x^2}.}$