Reduction of Order

MAT 275

Given a solution of a linear, homogeneous second-order ODE, it is possible to find another solution using a technique called **reduction of order**. The coefficients of y or its derivatives do not necessarily have to be constants.

Assume that the ODE has the form $a(x)y'' + b(x)y' + c(x)y = 0$ and that $y_1(x)$ is a solution.

We define the other solution $y_2(x) = v(x)y_1(x)$, where $v(x)$ is to be determined. Using the product rule, we find ${y_2}'(x)$ and ${y_2}''(x)$:

$$y_2'(x) = v(x)y_1'(x) + v'(x)y_1(x),$$

$$y_2''(x) = v(x)y_1''(x) + 2v'(x)y_1'(x) + v''(x)y_1(x).$$

These are substituted into the differential equation and simplified (next slide)

For the sake of space, we write $y_2 = vy_1$, $y_2' = vy_1' + v'y_1$ and $y_2'' = vy_1'' + 2v'y_1' + vy_1''$:

$$a(x)(vy_1'' + 2v'y_1' + v''y_1) + b(x)(vy_1' + v'y_1) + c(x)vy_1 = 0$$

Distribute to clear parentheses and regroup according to degrees of v:

$$v''[a(x)y_1] + v'[2a(x)y_1' + b(x)y_1] + v[\underbrace{a(x)y_1'' + b(x)y_1' + c(x)y_1}_{0}] = 0$$

Notice that the expression multiplied to v is 0: y_1 is a solution of the original differential equation so this expression is 0. We are left with

$$v''[a(x)y_1] + v'[2a(x)y_1' + b(x)y_1] = 0.$$

Replacing v' with u (and v'' with u'), we have a first-order ODE. Thus, we have reduced the order, and can be solved using separation of variables or integration factors.

$$u'[a(x)y_1] + u[2a(x)y_1' + b(x)y_1] = 0.$$

Example: Let $y_1 = e^{2x}$ be one solution of $y'' - 4y' + 4y = 0$. Use reduction of order to find another solution.

Solution: Using the form $u'[a(x)y_1] + u[2a(x)y_1' + b(x)y_1] = 0$, we have $a(x) = 1$, $b(x) = -4$, $y_1 = e^{2x}$ and $y_1' = 2e^{2x}$. Thus, we substitute:

$$u'[(1)(e^{2x})] + u[2(1)(2e^{2x}) + (-4)(e^{2x})] = 0.$$

In this example, the expression $2(1)(2e^{2x}) + (-4)(e^{2x}) = 0$, so we have

$$e^{2x}u' = 0.$$

Thus, $u' = 0$, so integrating, $u = k_1$, a constant. But since $v' = u$, we integrate again to find v, getting $v = k_1x + k_2$. Since $y_2 = vy_1$, we have $y_2 = (k_1x + k_2)e^{2x}$.

Since $y_1 = e^{2x}$ and $y_2 = (k_1 x + k_2)e^{2x}$ are solutions of $y'' - 4y' + 4y = 0$, the general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 e^{2x} + C_2(k_1 x + k_2)e^{2x}.$$

Clearing parentheses, we have

$$y = C_1 e^{2x} + C_2 k_1 x e^{2x} + C_2 k_2 e^{2x}.$$

We can combine the first and third term, calling $C_1 + C_2 k_2$ as "new" C_1, and $C_2 k_1$ as "new" C_2. Thus, the general solution can be written

$$y = C_1 e^{2x} + C_2 x e^{2x}.$$

You may recall that the auxiliary polynomial, $r^2 - 4r + 4 = 0$ has root $r = 2$, multiplicity 2. One solution is $y_1 = e^{2x}$ and the other $y_2 = xe^{2x}$. This process "justifies" that extra *x*.

Example: Given that $y_1 = t^2$ is one solution of $t^2 y'' + \frac{t}{2} y' - 3y = 0$. Find another solution y_2 of this differential equation, show that y_1 and y_2 are linearly independent, and state the general solution.

Solution: Set $y_2 = vt^2$, keeping in mind that v represents some function $v(t)$, and that $u(t) = v'(t)$. Using the form $u'[a(x)y_1] + u[2a(x)y_1' + b(x)y_1] = 0$, we have $a(x) = t^2$, $b(x) = \frac{t}{2}$, $y_1 = t^2$ and $y_1' = 2t$. Substituting, we have

$$u'[(t^2)(t^2)] + u\left[2(t^2)(2t) + \left(\frac{t}{2}\right)(t^2)\right] = 0.$$

Simplifying, we have

$$t^4 u' + \frac{9}{2} t^3 u = 0.$$

From the last screen, we have $t^4 u' + \frac{9}{2} t^3 u = 0$.

Divide through by t^4:

This declaration that *t* cannot be 0 is important when we get to the Wronskian step.

$$u' + \frac{9}{2t} u = 0, \qquad t \neq 0.$$

This solves using an integration factor: $\mu(t) = e^{\int \frac{9}{2t} dt} = e^{\frac{9}{2} \ln t} = e^{\ln t^{9/2}} = t^{9/2}$.

Thus, $u(t) = \frac{\int (t^{9/2})(0) dt + C}{t^{9/2}} = C t^{-9/2}$.

But remember, we want $v(t)$, where $v'(t) = u(t)$. Thus, integrate once to find $v(t)$:

$$v(t) = \int C t^{-9/2} \, dt = -\frac{2}{7} C t^{-7/2} + D.$$

Since $v(t) = -\frac{2}{7}Ct^{-7/2} + D$, then

$$y_2(t) = v(t)y_1(t) = \left(-\frac{2}{7}Ct^{-7/2} + D\right)t^2 = -\frac{2}{7}Ct^{-3/2} + Dt^2.$$

The general solution is

$$y = C_1y_1(t) + C_2y_2(t) = C_1t^2 + C_2\left(-\frac{2}{7}Ct^{-3/2} + Dt^2\right)$$
$$= C_1t^2 - \frac{2}{7}C_2Ct^{-3/2} + C_2Dt^2.$$

We can combine $C_1t^2 + C_2Dt^2$ as C_1t^2, where "new" $C_1 = C_1 + C_2D$, and we can let "new" $C_2 = -\frac{2}{7}C_2C$, so that the general solution is (probably):

$$y = C_1t^2 + C_2t^{-3/2}.$$

Let's make sure that $y_2 = t^{-3/2}$ solves $t^2 y'' + \frac{t}{2} y' - 3y = 0$. Taking derivatives, we have $y_2' = -\frac{3}{2} t^{-5/2}$ and $y_2'' = \frac{15}{4} t^{-7/2}$. Substituting then simplifying, we have

$$t^2 \left(\frac{15}{4} t^{-7/2} \right) + \frac{t}{2} \left(-\frac{3}{2} t^{-5/2} \right) - 3\left(t^{-3/2}\right) = 0$$

$$\frac{15}{4} t^{-3/2} - \frac{3}{4} t^{-3/2} - 3t^{-3/2} = 0$$

$$t^{-3/2} \left(\frac{15}{4} - \frac{3}{4} - 3 \right) = 0 \ldots$$ It works.

Next slide, we'll check they are linearly independent.

We now show that $y_1 = t^2$ and $y_2 = t^{-3/2}$ are linearly independent:

$$W\left(t^2, t^{-3/2}\right) = det\begin{bmatrix} t^2 & t^{-3/2} \\ 2t & (-3/2)t^{-5/2} \end{bmatrix}$$
$$= -\frac{3}{2}t^{-1/2} - 2t^{-1/2}$$
$$= -\frac{7}{2}t^{-1/2}.$$

Recall that three slides ago, we had to declare that $t \neq 0$ because we needed to divide through by t^4. Thus, since $t \neq 0$, the expression $-\frac{7}{2}t^{-1/2}$ cannot be 0. This means that the two functions $y_1 = t^2$ and $y_2 = t^{-3/2}$ are linearly independent and that

$$y = C_1 t^2 + C_2 t^{-3/2} \text{ is the general solution of } t^2 y'' + \frac{t}{2} y' - 3y = 0.$$