First-Order Autonomous Differential Equations

These differential equations are often solved using Separation of Variables. They are useful for modeling growth behavior, where the rate of growth is proportional in some manner to the quantity present, usually via a proportion.

Example 5.1: The rate of change of a population of a city is proportional to the population itself. If the population in 2010 was 25,000, and in 2018 was 32,000, forecast the population in 2025.

Solution: Let $P(t)$ be the population after t years, where $t = 0$ represents the year 2010. Note that $P > 0$. Time t could be negative, for example, if we wanted to estimate the city's population in 2005. Translated, we obtain

"The rate of change of a population":	$\frac{dP}{dt}$
"is"	$=$
"proportional to the population itself":	$kP.$

The number k is the proportionality constant. It is not a constant of integration. Also, when $t = 0$, the population is 25,000, and when $t = 8$, the population is 32,000. Assembling this into an equation, we have

$$\frac{dP}{dt} = kP, \quad \text{where} \quad P(0) = 25{,}000 \quad \text{and} \quad P(8) = 32{,}000.$$

Separate the variables. Note that it makes sense for $P > 0$ in this example:

$$\frac{dP}{P} = k\,dt.$$

Integrate and solve for P:

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln P = kt + C$$

$$P = e^{kt+C}$$

$$P = e^{kt}e^{C}$$

$$P(t) = Ce^{kt}.$$

Use the ordered pair (0, 25000) to determine *C*:

$$25{,}000 = Ce^{0t}$$
$$C = 25{,}000.$$

Thus, we now have

$$P(t) = 25{,}000e^{kt}.$$

Use the ordered pair (8, 32000) to determine *k*:

$$32{,}000 = 25{,}000e^{k(8)}$$
$$\frac{32}{25} = e^{8k}$$
$$\ln\left(\frac{32}{25}\right) = 8k$$
$$k = \frac{1}{8}\ln\left(\frac{32}{25}\right) \approx 0.031.$$

The specific solution that meets all stated conditions is

$$P(t) = 25{,}000e^{0.031t}.$$

In 2025, $t = 15$, so we have

$$P(15) = 25{,}000e^{0.031(15)} \approx 39{,}800.$$

In 2025, there will be about 39,800 people in the city.

Example 5.2: The rate of change in the value of a stock is inversely proportional to the square of the value of that stock. If the stock's value was \$20 at noon, and was \$23 at 3 p.m., what is the stock's value at 5 p.m.?

Solution: Let $V(t)$ be the value, in dollars, of the stock t hours after noon (when $t = 0$). Translating into mathematics, we have

"The rate of change in the value…"	$\frac{dV}{dt}$
"is"	$=$
"inversely proportional to the square of the value…"	$\frac{k}{V^2}$

The known conditions are $V(0) = \$20$ and $V(3) = \$23$. Thus, the differential equation that models this growth is

$$\frac{dV}{dt} = \frac{k}{V^2} \quad \text{where} \quad V(0) = 20 \quad \text{and} \quad V(3) = 23.$$

Separate the variables:

$$V^2\, dV = k\, dt$$

Integrate:

$$\int V^2\, dV = \int k\, dt\,, \quad \text{which gives} \quad \frac{1}{3}V^3 = kt + C.$$

Multiply both sides by 3 to clear fractions:

$$V^3 = 3kt + C.$$

The constant of integration *C* is a generic, so 3*C* is the same as writing *C*. Take the cube root:

$$V(t) = \sqrt[3]{3kt + C}.$$

This is the general model that governs the stock's value.

To find *C*, use the condition, $V(0) = 20$:

$$20 = \sqrt[3]{3k(0) + C}$$

$$20 = \sqrt[3]{C}$$

$$C = 20^3 = 8{,}000.$$

We now have

$$V(t) = \sqrt[3]{3kt + 8{,}000}.$$

To find k, use the other condition, $V(3) = 23$:

$$23 = \sqrt[3]{3k(3) + 8{,}000}.$$

$$23^3 = 9k + 8{,}000$$

$$12{,}167 = 9k + 8{,}000$$

$$4{,}167 = 9k$$

$$k = \frac{4{,}167}{9} = 463.$$

The specific solution is now

$$V(t) = \sqrt[3]{3(463)t + 8{,}000} = \sqrt[3]{1{,}389t + 8{,}000}.$$

In this case, it's permissible to combine the factors 3 and k in front of the t. It won't affect the later calculation.

The stock's value at 5 p.m. means $t = 5$:

$$V(5) = \sqrt[3]{1{,}389(5) + 8{,}000} \approx \$24.63.$$

Example 5.3: Find the general solution of $y' = y^2 + y$.

Solution: Separating variables gives

$$\frac{dy}{y^2 + y} = dx, \qquad y \neq 0, -1.$$

Before antidifferentiating the left side, the denominator needs to be written as the sum of smaller fractions, using a process called partial fraction decomposition:

$$\frac{1}{y^2 + y} = \frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1},$$

where A and B are the unknown numerators of the smaller fractions. The two fractions are then recomposed by finding the common denominator:

$$\frac{1}{y(y+1)} = \frac{A(y+1) + By}{y(y+1)}.$$

The numerators are now compared. On the right side, clear parentheses and reorder the terms according to powers of y:

$$1 = (A + B)y + A.$$

By viewing the left side as $0y + 1$, relate the expressions on the right to those on the left. Thus, $A + B = 0$ and $A = 1$. This forces $B = -1$. The partial fraction decomposition is complete, and we have

$$\frac{1}{y^2 + y} = \frac{1}{y} - \frac{1}{y+1}.$$

This is now in a form to be antidifferentiated:

$$\int \frac{dy}{y^2 + y} = \int dx$$
$$\int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = \int dx$$
$$\ln y - \ln(y+1) = x + C.$$

To solve for y, use the logarithm property $\ln a - \ln b = \ln \frac{a}{b}$:

$$\ln\left(\frac{y}{y+1}\right) = x + C.$$

This is rewritten using base-e notation. Note that $e^{x+C} = e^x e^C = Ce^x$. Then y is isolated through algebra:

$$\frac{y}{y+1} = Ce^x$$
$$y = Ce^x(y+1)$$
$$y = Cye^x + Ce^x$$
$$y - Cye^x = Ce^x$$
$$y(1 - Ce^x) = Ce^x$$
$$y = \frac{Ce^x}{1 - Ce^x}.$$

While this is correct, a simpler form can be found by dividing the numerator and denominator by Ce^x:

$$\frac{Ce^x/Ce^x}{(1 - Ce^x)/Ce^x} = \frac{1}{\left(\frac{1}{Ce^x}\right) - 1}.$$

The expression $\frac{1}{Ce^x} = Ce^{-x}$ by treating $\frac{1}{C}$ as "new" C. The simplified form of the general solution is

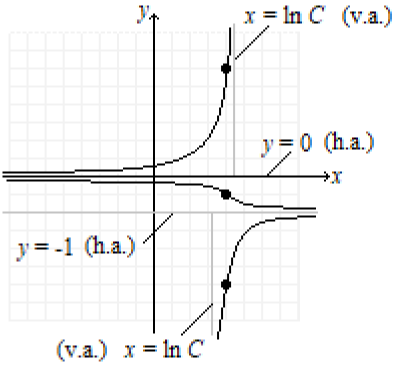
$$y = \frac{1}{Ce^{-x} - 1}, \qquad x \neq \ln C.$$

Let's explore the behavior of the solution curves, each dependent on an initial condition (a, b). Direct evaluation of the initial condition in the general solution shows that $C = \frac{1+b}{b}e^a$. When $b < -1$ or $b > 0$, then C is positive, and when $-1 < b < 0$, then C is negative.

The range of a solution curve will be in the interval $y < -1$, $-1 < y < 0$ or $y > 0$, where C is positive when $y < -1$ or $y > 0$, and C is negative when $-1 < y < 0$. Three cases emerge:

If $y < -1$, then C is positive, but the denominator $Ce^{-x} - 1$ must be negative for the whole expression $1/(Ce^{-x} - 1)$ to be negative, thus $Ce^{-x} - 1 < 0$. Isolating x, we find that the domain will be in the interval $x > \ln C$.

If $-1 < y < 0$, then the denominator must also be negative, but that since C is negative also, the expression $\ln C$ is not defined. In other words, the domain of x is the whole Real line.

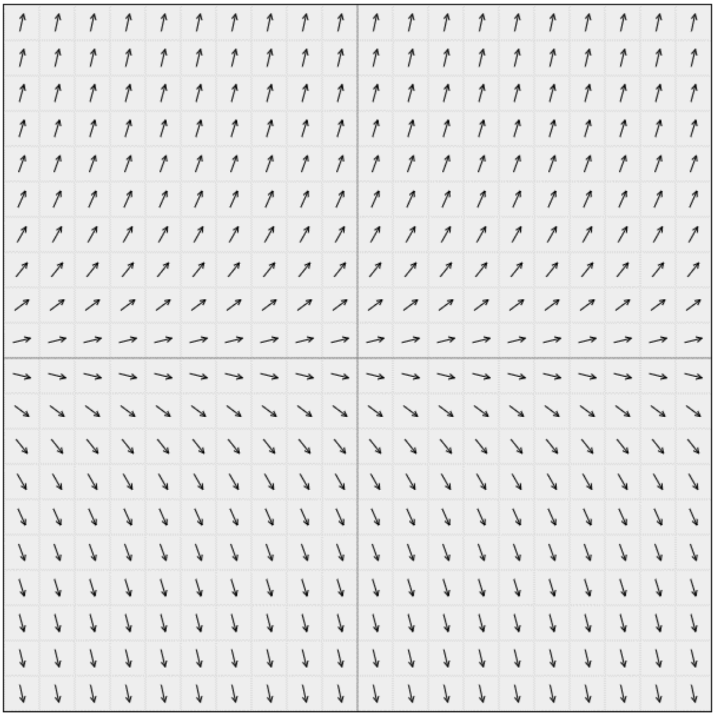


If $y > 0$, then C is positive, and the denominator must be positive for the whole expression $1/(Ce^{-x} - 1)$ to be positive, thus $Ce^{-x} - 1 > 0$. Isolating x, we find that the domain will be in the interval $x < \ln C$.

In the image, the asymptotes show how the xy-plane is divided into potential solution regions. The initial conditions for this illustration are (2,3), (2, –1/2) and (2, –3). For the initial condition (2,3), $C = \frac{4}{3}e^2$ so that $\ln\left(\frac{4}{3}e^2\right) \approx 2.288$ is the upper bound of the domain, and for (2, –3), $C = \frac{2}{3}e^2$ so that $\ln\left(\frac{2}{3}e^2\right) \approx 1.595$ is the lower bound of the domain. Each is a vertical asymptote for its respective solution curve.

Direction Fields for First-Order Autonomous Equations

For an autonomous differential equation, the *x* is absent, so the slopes only depend on the y value at that point. For each y-value, the slope at each point is the same, going across horizontally. For example, the direction field for $y' = y$ is



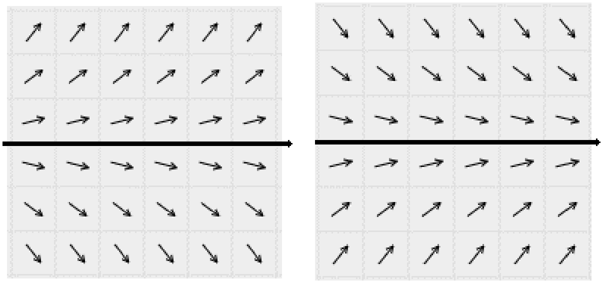
Scale: each gridline is 0.5 unit. Courtesy https://aeb019.hosted.uark.edu/dfield.html.

The solution curves will trend away from the *x*-axis (*y* = 0), either entirely above the *x*-axis or entirely below it, depending on the initial condition. No solution curve for this differential equation will cross (or touch) the *x*-axis. Recall that the general solution for $y' = y$ is $y = Ce^x$. Picking any point in the plane (except along the *x*-axis), and following the arrows, one sees the familiar shape of the exponential function.

In an autonomous differential equation, the y values for which $y' = 0$ are called **equilibrium** solutions. On a direction field, equilibrium solutions will be where the slope lines are horizontal, reading across left to right. There are three types of equilibrium solutions:

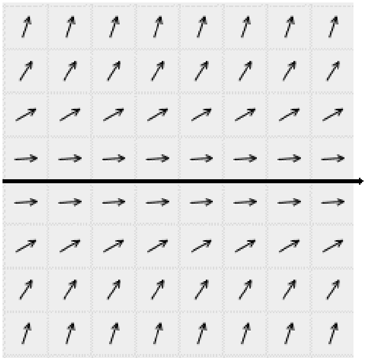
If the solution curves trend away from the equilibrium both above and below as *x* increases, it is an **unstable** equilibrium.

If the solution curves trend toward the equilibrium asymptotically both above and below as *x* increases, it is a **stable** equilibrium.



Left: example of an unstable equilibrium
Right: Example of a stable equilibrium

If the solution curves trend away from the equilibrium on one side of the equilibrium, and trend toward the equilibrium on the other side as *x* increases, it is a **semistable** equilibrium.



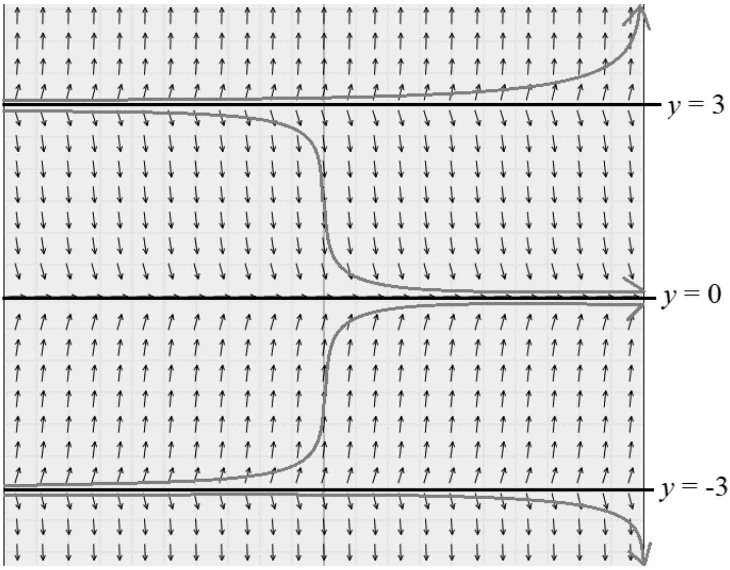
Example of a semistable equilibrium.
The curves trend away from equilibrium when $y > 0$, and trend toward equilibrium when $y < 0$.

Example 5.4: Given $y' = y^3 - 9y$. Find all equilibrium solutions, and determine if they are stable, unstable or semistable.

Solution: The equilibrium points are where $y' = y^3 - 9y = 0$. Factoring, we have $y(y+3)(y-3) = 0$, so the equilibrium solutions are where $y = 0$, $y = 3$, and $y = -3$. This divides the *y*-axis into four intervals. A value is chosen within each interval and evaluated to determine the sign of the slope:

$y > 3$:	Choose $y = 4$.	Thus, $y' = 4^3 - 9(4) > 0$.
$0 < y < 3$:	Choose $y = 1$.	Thus, $y' = 1^3 - 9(1) < 0$.
$-3 < y < 0$:	Choose $y = -1$.	Thus, $y' = (-1)^3 - 9(-1) > 0$.
$y < -3$:	Choose $y = -4$.	Thus, $y' = (-4)^3 - 9(-4) < 0$.

Curves above $y = 3$ slope upward, away from $y = 3$, and curves between $y = 0$ and $y = 3$ will curve down, also away from $y = 3$. So $y = 3$ is unstable. By similar reasoning, $y = 0$ is stable and $y = –3$ is unstable.



Direction field showing stable equilibrium at $y = 0$, and unstable equilibrium at $y = \pm 3$.
Courtesy https://aeb019.hosted.uark.edu/dfield.html.