Laplace Transforms of Periodic Functions

MAT 275

Laplace Transform of Periodic Functions

A function f is **periodic** with period T if $f(t+T) = f(t)$ for all t, where T is the smallest non-zero value for which $f(t+T) = f(t)$.

For example, $f(t) = \cos(t)$ is periodic with periods $n\pi$, where n is an even integer. The smallest such period is 2π.

Suppose $y = f(t)$ is periodic with period T. Apply the Laplace Transform operator:

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt\,.$$

Write the integral as a sum of two integrals, from $0 \le t < T$ and $T \le t < \infty$:

$$\int_0^\infty f(t)e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + \int_T^\infty f(t+T)e^{-st}\,dt\,.$$

From the last slide, we have $\int_0^\infty f(t)e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + \int_T^\infty f(t+T)e^{-st}\,dt$.

In the second integral, substitute $u = t + T$, so that $t = u - T$. Note that $du = dt$ (since T is a constant) and that with this shift, the integral in u has a lower bound of 0:

$$\int_0^\infty f(t)e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + \int_0^\infty f(u)e^{-s(u-T)}\,du.$$

Observe that $e^{-s(u-T)} = e^{-su}e^{-sT}$. Bring the e^{-sT} outside as it is constant with respect to u:

$$\int_0^\infty f(t)e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + e^{-sT}\int_0^\infty f(u)e^{-su}\,du.$$

Both $\int_0^\infty f(t)e^{-st}\,dt$ and $\int_0^\infty f(u)e^{-su}\,du$ are identical integrals since the variables of integration are dummy variables. Thus, we have… (next slide)

From the last slide, we have $\int_0^\infty f(t)e^{-st}\,dt = \int_0^T f(t)e^{-st}\,dt + e^{-sT}\int_0^\infty f(u)e^{-su}\,du$.

Replace the first and last integrals with $L\{f(t)\}$:

$$L\{f(t)\} = \left(\int_0^T f(t)e^{-st}\,dt\right) + e^{-sT}L\{f(t)\}.$$

Solve for $L\{f(t)\}$:

$$L\{f(t)\} - e^{-sT}L\{f(t)\} = \int_0^T f(t)e^{-st}\,dt$$

$$L\{f(t)\}(1 - e^{-sT}) = \int_0^T f(t)e^{-st}\,dt$$

$$L\{f(t)\} = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-sT}}$$

Example: Find $L\{\sin(t)\}$ using the formula $L\{f(t)\} = \frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}}$.

We already know that $L\{\sin(bt)\} = b/(s^2+b^2)$ so that we should get $L\{\sin(t)\} = 1/(s^2 + 1)$.

Solution: The sine function has period $T = 2\pi$, so we have

$$L\{\sin(t)\} = \frac{\int_0^{2\pi} \sin(t)\, e^{-st}\, dt}{1 - e^{-2\pi s}}.$$

The integral $\int_0^{2\pi} \sin(t)\, e^{-st}\, dt$ is evaluated using integration-by-parts. We have

$$\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \left[-\frac{1}{s}\sin(t)\, e^{-st}\right]_0^{2\pi} + \frac{1}{s}\int_0^{2\pi} \cos(t)\, e^{-st}\, dt\,.$$

The term $\left[-\frac{1}{s}\sin(t)\, e^{-st}\right]_0^{2\pi} = 0$ after evaluation at the bounds so it drops out.

So we have $\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \frac{1}{s}\int_0^{2\pi} \cos(t)\, e^{-st}\, dt$.

Integrate by parts again on the right-most integral:

$$\int_0^{2\pi} \cos(t)\, e^{-st}\, dt = \left[-\frac{1}{s} e^{-st} \cos(t)\right]_0^{2\pi} - \frac{1}{s}\int_0^{2\pi} \sin(t)\, e^{-st}\, dt.$$

The term $\left[-\frac{1}{s} e^{-st} \cos(t)\right]_0^{2\pi} = -\frac{1}{s} e^{-2\pi s} + \frac{1}{s}$ after evaluating the bounds. Making substitutions, we have

$$\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \frac{1}{s}\left[\left(-\frac{1}{s} e^{-2\pi s} + \frac{1}{s}\right) - \frac{1}{s}\int_0^{2\pi} \sin(t)\, e^{-st}\, dt\right].$$

The rest is just algebra… (next slide)

We have $\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \frac{1}{s}\left[\left(-\frac{1}{s}e^{-2\pi s} + \frac{1}{s}\right) - \frac{1}{s}\int_0^{2\pi} \sin(t)\, e^{-st}\, dt\right]$. Distribute:

$$\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \left(-\frac{1}{s^2}e^{-2\pi s} + \frac{1}{s^2}\right) - \frac{1}{s^2}\int_0^{2\pi} \sin(t)\, e^{-st}\, dt\,.$$

Collect the two integrals to the left side.

$$\int_0^{2\pi} \sin(t)\, e^{-st}\, dt + \frac{1}{s^2}\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \frac{1-e^{-2\pi s}}{s^2}$$

Factor the integral, and simplify the coeffs as a single rational expression.

$$\left(\frac{s^2+1}{s^2}\right)\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \frac{1-e^{-2\pi s}}{s^2}$$

Thus,

$$\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \frac{1-e^{-2\pi s}}{s^2}\cdot\frac{s^2}{s^2+1} = \frac{1-e^{-2\pi s}}{s^2+1}.$$

Recall that

$$L\{\sin(t)\} = \frac{\int_0^{2\pi} \sin(t)\, e^{-st}\, dt}{1 - e^{-2\pi s}}.$$

From the last slide, we found that

$$\int_0^{2\pi} \sin(t)\, e^{-st}\, dt = \frac{1 - e^{-2\pi s}}{s^2 + 1}.$$

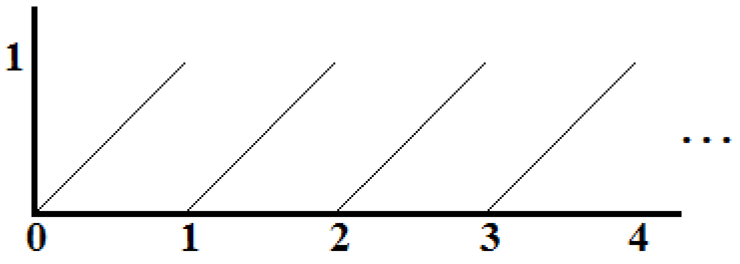
Thus,

$$L\{\sin(t)\} = \frac{1}{1 - e^{-2\pi s}} \left(\frac{1 - e^{-2\pi s}}{s^2 + 1} \right) = \frac{1}{s^2 + 1}.$$

See, it works!

Example: Find the Laplace Transform of the sawtooth function shown below:

It is given by $f(t) = t$ for $0 \le t < 1$
and $f(t-1) = f(t)$ for $t \ge 1$.



Solution: The period is $T = 1$, so we have $L\{f(t)\} = \frac{\int_0^1 te^{-st}\,dt}{1-e^{-s(1)}}$.

The integral is evaluated using integration-by-parts:

$$\int_0^1 te^{-st}\,dt = \left[-\frac{t}{s}e^{-st}\right]_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt = \frac{1}{s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2} = \frac{1-e^{-s}-se^{-s}}{s^2}.$$

Thus,

$$L\{f(t)\} = \frac{\int_0^1 te^{-st}\,dt}{1-e^{-s(1)}} = \frac{1-e^{-s}-se^{-s}}{s^2}\cdot\frac{1}{1-e^{-s}} = \frac{1-e^{-s}(s+1)}{s^2(1-e^{-s})}.$$