Matrix Review: Determinants, Eigenvalues & Eigenvectors

MAT 275

A **linear system** is two or more linear equations in two or more variables taken together.

For example, $\begin{matrix} 3x + 2y = 11 \\ -x + 5y = 19 \end{matrix}$ is a system of two linear equations in two variables.

A **solution of a system** is any ordered pair (triple, *etc.*) that solves all equations of the system simultaneously.

For example, $(1,4)$ is a solution of the above system, since $3(1) + 2(4) = 3 + 8 = 11$ is true, and $-(1) + 5(4) = -1 + 20 = 19$ is also true.

A system is **consistent** if it has at least one solution. Otherwise, the system is **inconsistent**. The above system is consistent.

Linear systems may have no solution (inconsistent), one solution or infinitely many solutions.

Matrices (singular: Matrix)

Matrices are used to solve systems of linear equations as well as to give insight as to the structure of the system. There are many ways to solve a linear system using matrices. Some that you may have seen are to use Gaussian Row Operations (i.e. the RREF method), Cramer's Rule, and so on.

For example, the system $\begin{matrix} 3x+2y=11 \\ -x+5y=19 \end{matrix}$ can be written as an equation of matrices:

$$\underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 11 \\ 19 \end{bmatrix}}_{B}.$$

Matrix A is the coefficient matrix, X is the variable matrix, and B is the constants matrix. Since the system has two equations in two variables, A is a "2×2" matrix, or a **square** matrix. Matrices X and B are called **vectors** since they each have just one column.

All square matrices have associated with each a unique real number called its **determinant**. For the 2×2 case, the formula is:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

Here, "det" stands for determinant. It acts like a function that assigns to each square matrix a number, where that number is given by the above formula.

Examples:

$$\det \begin{bmatrix} 2 & 6 \\ 3 & 7 \end{bmatrix} = (2)(7) - (3)(6) = 14 - 18 = -4.$$

$$\det \begin{bmatrix} 4 & 2 \\ -5 & 6 \end{bmatrix} = (4)(6) - (-5)(2) = 24 - (-10) = 34.$$

$$\det \begin{bmatrix} 3 & 6 \\ 12 & 24 \end{bmatrix} = (3)(24) - (12)(6) = 72 - 72 = 0.$$

If the determinant of a square matrix is 0, that matrix is called **singular**.

The system $\begin{aligned} 3x + 2y &= 11 \\ -x + 5y &= 19 \end{aligned}$ can be written as $\underbrace{\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 11 \\ 19 \end{bmatrix}}_{B}$.

Matrices X and B are both 2×1 in size (2 rows, 1 column). **Matrix multiplication** is defined as the linear combinations of rows of the left factor with columns of the right factor. Thus, if $x = 1$ and $y = 4$, we have

$$\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} (3)(1) + (2)(4) \\ (-1)(1) + (5)(4) \end{bmatrix} = \begin{bmatrix} 11 \\ 19 \end{bmatrix}.$$

Matrices can also be multiplied by a constant, called a **scalar**. For example:

$$2 \cdot A = 2 \cdot \begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 & 2 \cdot 2 \\ 2 \cdot (-1) & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -2 & 10 \end{bmatrix}.$$

$$7 \cdot B = 7 \cdot \begin{bmatrix} 11 \\ 19 \end{bmatrix} = \begin{bmatrix} 7 \cdot 11 \\ 7 \cdot 19 \end{bmatrix} = \begin{bmatrix} 77 \\ 133 \end{bmatrix}.$$

Eigenvalues and Eigenvectors

Something interesting happens with square matrices and specially-chosen scalar multiples and vectors. For example, let $A = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$, and let vectors $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Note what happens when we calculate $A \cdot v_1$ and $A \cdot v_2$:

$$A \cdot v_1 = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \qquad \text{which is the same as} \qquad 2 \cdot v_1 = 2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

$$A \cdot v_2 = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}, \qquad \text{which is the same as} \qquad 6 \cdot v_2 = 6 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}.$$

In other words, for specially-chosen vectors, the action of multiplying it by a matrix A has the same result as if that vector were multiplied by some non-zero scalar constant.

In these cases, we call 2 and 6 the **eigenvalues** of A, and call the vectors v_1 and v_2 **eigenvectors** of A. Eigenvalues are usually denoted by the Greek letter lambda, λ.

Finding Eigenvalues and Eigenvectors

Let A be a square matrix (we'll look at the 2×2 case for now). We seek to find a scalar eigenvalue λ and a non-zero eigenvector v such that

$$A \cdot v = \lambda \cdot v.$$

We need to write the above equation so that the right side "looks like" the left side. By writing $\lambda = \lambda \cdot I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, then both sides are essentially a 2×2 matrix multiplying to a 2×1 matrix. This allows us to move the terms around algebraically:

$$A \cdot v = \lambda \cdot I \cdot v \qquad \text{is the same as} \qquad A \cdot v - \lambda \cdot I \cdot v = 0.$$

Factoring, we have $(A - \lambda I)v = 0$. Since $v \neq 0$, then this is only true when $\det(A - \lambda I) = 0$. This is the formula we use to find eigenvalues.

Example: Find the eigenvalues of $A = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$.

Solution: Since $\lambda \cdot I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, we have

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 1 \\ 3 & 5-\lambda \end{bmatrix} = (3-\lambda)(5-\lambda) - 3.$$

This is set to 0 and solved for λ:

$$\begin{gathered} (3-\lambda)(5-\lambda) - 3 = 0 \\ \lambda^2 - 8\lambda + 15 - 3 = 0 \\ \lambda^2 - 8\lambda + 12 = 0 \\ (\lambda - 2)(\lambda - 6) = 0 \\ \lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 6. \end{gathered}$$

The two eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 6$, where the subscripts help us keep track of each eigenvalue. Now we need to find their eigenvectors.

From the last slide, we found the two eigenvalues of $A = \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}$ to be $\lambda_1 = 2$ and $\lambda_2 = 6$.

To find the eigenvectors, use a generic vector $v = \begin{bmatrix} a \\ b \end{bmatrix}$, and solve $(A - \lambda I)v = 0$. We'll start with $\lambda_1 = 2$:

$$\begin{bmatrix} 3-2 & 1 \\ 3 & 5-2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

Note that the square matrix is singular. This always happens after subtracting the diagonal elements by the eigenvalue and is a good check of your work.

The top row multiplies to $a + b = 0$. If we choose $a = 1$, then that forces $b = -1$, so then we have an eigenvector associated with eigenvalue $\lambda_1 = 2$: it is $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Note that subscripts are used to "tie" the eigenvalue and eigenvector together.

Now let's find the eigenvector for $\lambda_2 = 6$:

$$\begin{bmatrix} 3-6 & 1 \\ 3 & 5-6 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = 0.$$

The top row multiplies to $-3a + b = 0$. If we choose $a = 1$, then that forces $b = 3$, so then we have an eigenvector associated with eigenvalue $\lambda_2 = 6$: it is $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

We'll be using eigenvalues and eigenvectors to solve systems of differential equations. We can use them to show direction fields and gain insight as to how the solutions of a system behave.

Example: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$.

Solution: We have $\det \begin{bmatrix} 3-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = 0$, which gives $(3-\lambda)(1-\lambda) = 0$. Thus, the two eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$. (It makes no difference the order of the subscripts.)

The eigenvector for $\lambda_1 = 3$ is $v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} 3-3 & 1 \\ 0 & 1-3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This simplifies to $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Note that $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ is singular.

Multiplying the top row by the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$, we have $0a + 1b = 0$, or simply $b = 0$. However, eigenvector v_1 cannot be a zero-vector. Thus, *any* other value may be chosen for a. We'll let $a = 1$. Thus, an eigenvector of $\lambda_1 = 3$ is $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution, continued: Now we find the eigenvector for $\lambda_2 = 1$.

The eigenvector for $\lambda_2 = 1$ is $v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} 3-1 & 1 \\ 0 & 1-1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This simplifies to $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Again, note that $\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$ is singular.

Multiplying the top row by the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$, we have $2a + 1b = 0$. Letting $a = 1$, this forces $b = -2$. Thus, the eigenvector of $\lambda_2 = 1$ is $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Comment: Any other vector that is dependent to the eigenvector (except the zero vector) is acceptable. For example, we have $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Other acceptable eigenvectors v_2 are $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}, \begin{bmatrix} 10 \\ -20 \end{bmatrix}$, and so on. The same is true for $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Sometimes, the eigenvalues are not "convenient" integers…

Example: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

Solution: We have $\det\begin{bmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} = 0$, which gives $\lambda^2 - 4\lambda + 1 = 0$. Using the quadratic formula, we have $\lambda_1 = 2 + \sqrt{3}$ and $\lambda_2 = 2 - \sqrt{3}$.

The eigenvector of $\lambda_1 = 2 + \sqrt{3}$ is $v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} 2-(2+\sqrt{3}) & 1 \\ 3 & 2-(2+\sqrt{3}) \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This simplifies to $\begin{bmatrix} -\sqrt{3} & 1 \\ 3 & -\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Is $\begin{bmatrix} -\sqrt{3} & 1 \\ 3 & -\sqrt{3} \end{bmatrix}$ singular? Its determinant is $(-\sqrt{3})(-\sqrt{3}) - 3 = 3 - 3 = 0$, so yes, it is singular. Multiplying the top row by $\begin{bmatrix} a \\ b \end{bmatrix}$ gives $-\sqrt{3}a + b = 0$. Letting $a = 1$, then $b = \sqrt{3}$, and an eigenvector of $\lambda_1 = 2 + \sqrt{3}$ is $v_1 = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, or any non-zero multiple.

Solution (continued): Now we find the eigenvector of $\lambda_2 = 2 - \sqrt{3}$.

It is $v_2 = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} 2 - \left(2 - \sqrt{3}\right) & 1 \\ 3 & 2 - \left(2 - \sqrt{3}\right) \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This simplifies to $\begin{bmatrix} \sqrt{3} & 1 \\ 3 & \sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The matrix is singular (you should verify this).

Multiplying the top row by $\begin{bmatrix} a \\ b \end{bmatrix}$ gives $\sqrt{3}a + b = 0$. Letting $a = 1$, then $b = -\sqrt{3}$, and an eigenvector of $\lambda_2 = 2 - \sqrt{3}$ is $v_2 = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$, or any non-zero multiple.

Eigenvalues may not be real numbers…

Example: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$.

Solution: We have $\det \begin{bmatrix} 1-\lambda & -4 \\ 1 & 1-\lambda \end{bmatrix} = 0$, which simplifies to $\lambda^2 - 2\lambda + 5 = 0$. Using the quadratic formula, we have $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$.

The eigenvector for $\lambda_1 = 1 + 2i$ is $v_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} 1-(1+2i) & -4 \\ 1 & 1-(1+2i) \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This simplifies to $\begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. To be sure this is correct, we check it determinant: It is $(-2i)(-2i) - (1)(-4) = 4i^2 + 4 = -4 + 4 = 0$. It is singular!

The top row multiplied by $\begin{bmatrix} a \\ b \end{bmatrix}$ gives $-2ia - 4b = 0$. If $a = 1$, then $b = -\frac{1}{2}i$. Thus, an eigenvector for $\lambda_1 = 1 + 2i$ is $v_1 = \begin{bmatrix} 1 \\ -i/2 \end{bmatrix}$, or $\begin{bmatrix} 2 \\ -i \end{bmatrix}$, or any non-zero scalar multiple.

Solution, continued: Now we find an eigenvector for $\lambda_2 = 1 - 2i$.

It is $v_2 = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\begin{bmatrix} 1-(1-2i) & -4 \\ 1 & 1-(1-2i) \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This simplifies to $\begin{bmatrix} 2i & -4 \\ 1 & 2i \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. It is singular (you check this).

The top row multiplied by $\begin{bmatrix} a \\ b \end{bmatrix}$ gives $2ia - 4b = 0$. If $a = 1$, then $b = \frac{1}{2}i$. Thus, an eigenvector for $\lambda_2 = 1 - 2i$ is $v_2 = \begin{bmatrix} 1 \\ i/2 \end{bmatrix}$, or $\begin{bmatrix} 2 \\ i \end{bmatrix}$, or any non-zero scalar multiple.

Example: Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

Solution: We have $\det \begin{bmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{bmatrix} = 0$, which simplifies (in factored form) to $(2-\lambda)(1-\lambda)(3-\lambda) = 0$, so we have three eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 3$.

The eigenvector for $\lambda_1 = 2$ is a vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\begin{bmatrix} 2-2 & 1 & -1 \\ 0 & 1-2 & 2 \\ 0 & 0 & 3-2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

This simplifies to $\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The square matrix is singular, but now we must do a little extra simplification: we find its RREF equivalent, which gives… (next slide)

The square matrix from the last slide is simplified into its RREF equivalent, giving $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Multiplying, we get $b = 0$ and $c = 0$, but no restriction on a. Thus, we let a be any non-zero value. The eigenvector associated with $\lambda_1 = 2$ is $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, or any non-zero scalar multiple.

For $\lambda_2 = 1$, the square matrix in RREF is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. When multiplied, the top row gives $a + b = 0$ and the second row gives $c = 0$. From the top row, if we let $a = 1$, then $b = -1$, so an eigenvector for $\lambda_2 = 1$ is $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

For the third eigenvector, see next slide…

For $\lambda_3 = 3$, the square matrix in RREF is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. When multiplied, the top row gives $a = 0$ and the second row gives $b - c = 0$. From the second row, if we let $b = 1$, then $c = 1$, so an eigenvector for $\lambda_3 = 3$ is $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

In conclusion, the three eigenvalues of $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ are $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 3$ and the three eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.