Laplace Transforms: Special Cases

Derivative Rule, Shift Rule, Gamma Function & $f(ct)$ Rule

MAT 275

- **Derivative Rule:** If $L\{f(t)\} = H(s)$, then $L\{-tf(t)\} = H'(s)$.

Proof: Using the definition of the Laplace Transform, we have $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$.

Differentiate both sides with respect to s: $\frac{d}{ds}H(s) = \frac{d}{ds}L\{f(t)\} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt$.

The integrand can be differentiated "in place" with respect to s. In this step, $f(t)$ acts as a constant multiplier. Note that $\frac{d}{ds}e^{-st} = -te^{-st}$:

$$\frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty \frac{d}{ds}\left(f(t)e^{-st}\right)dt = \int_0^\infty -te^{-st}f(t)\,dt.$$

Thus, $H'(s) = \int_0^\infty -te^{-st}f(t)\,dt = L\{-tf(t)\},$ where $L\{f(t)\} = H(s)$.

Corollary: $H''(s) = L\{(-t)(-t)f(t)\} = L\{t^2 f(t)\}$. In general, $H^{(n)}(s) = L\{(-t)^n f(t)\}$.

Example: Find $L\{t\cos(t)\}$.

Solution: First, find $L\{\cos(t)\}$, which is $L\{\cos(t)\} = \frac{s}{s^2+1}$. Thus, $H(s) = \frac{s}{s^2+1}$.

Differentiating H, we have $H'(s) = \frac{1-s^2}{(s^2+1)^2}$.

Since $L\{-tf(t)\} = H'(s)$, then moving the negative, we have $L\{tf(t)\} = -H'(s)$.

Therefore,

$$L\{t\cos(t)\} = -\left(\frac{1-s^2}{(s^2+1)^2}\right) = \frac{s^2-1}{(s^2+1)^2}.$$

Example: Find $L\{te^{2t}\}$ and $L\{t^2e^{2t}\}$.

Solution: Start with $H(s) = L\{e^{2t}\} = \frac{1}{s-2}$. Differentiate twice, so that $H'(s) = -\frac{1}{(s-2)^2}$ and $H''(s) = \frac{2}{(s-2)^3}$.

So we have $L\{-te^{2t}\} = H'(s)$ or equivalently, $L\{te^{2t}\} = -H'(s) = -\left(-\frac{1}{(s-2)^2}\right) = \frac{1}{(s-2)^2}$.

Similarly, we have $L\{(-t)^2e^{2t}\} = L\{t^2e^{2t}\} = H''(s) = \frac{2}{(s-2)^3}$.

Example: Find $y = L^{-1}\left\{\frac{4}{(s+1)^3}\right\}$.

Solution: Integrate twice, so we have $\int \frac{4}{(s+1)^3}\,ds = -\frac{2}{(s+1)^2}$, and $\int \frac{-2}{(s+1)^2}\,ds = \frac{2}{s+1}$.

Thus, $L^{-1}\left\{\frac{2}{s+1}\right\} = 2e^{-t}$. Since $L\{t^2f(t)\} = H''(s)$, we have $y = L^{-1}\left\{\frac{4}{(s+1)^3}\right\} = 2t^2e^{-t}$.

Example: Use Laplace Transforms to solve $y'' - 6y' + 9y = 0, y(0) = 1, y'(0) = 0$.

Solution: We have

$$L\{y''\} - 6L\{y'\} + 9L\{y\} = L\{0\}$$
$$s^2L\{y\} - sy(0) - y'(0) - 6\big(sL\{y\} - y(0)\big) + 9L\{y\} = 0$$
$$s^2L\{y\} - s - 6sL\{y\} + 6 + 9L\{y\} = 0$$
$$L\{y\}[s^2 - 6y + 9] = s - 6$$
$$L\{y\} = \frac{s-6}{s^2 - 6s + 9}.$$

Thus, $y = L^{-1}\left\{\frac{s-6}{s^2-6s+9}\right\}$. Using partial fractions, we have $\frac{s-6}{s^2-6s+9} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$. The denominator has a linear factor multiplicity 2.

Solving for A and B, we get $\frac{s-6}{s^2-6s+9} = \frac{1}{s-3} - \frac{3}{(s-3)^2}$

The solution is $y = L^{-1}\left\{\frac{1}{s-3} - \frac{3}{(s-3)^2}\right\} = L^{-1}\left\{\frac{1}{s-3}\right\} + L^{-1}\left\{\frac{-3}{(s-3)^2}\right\}$.

The first term's inversion is $L^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$.

For the second, recognize that $\frac{-3}{(s-3)^2}$ is the derivative of $\frac{3}{s-3}$. Using the rule $L\{-te^{2t}\} = H'(s)$, we have $L^{-1}\left\{\frac{3}{s-3}\right\} = 3e^{3t}$, so that $L^{-1}\left\{\frac{-3}{(s-3)^2}\right\} = -3te^{3t}$. Remember to attach a negative because of the leading negative in $L\{-te^{2t}\}$.

The solution of $y'' - 6y' + 9y = 0, y(0) = 1, y'(0) = 0$ is

$$y = e^{3t} - 3te^{3t}.$$

- **Shift Rule:** If $L\{f(t)\} = H(s)$, then $L\{e^{at}f(t)\} = H(s-a)$.

Proof: We have $L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$, so that

$$L\{e^{at}f(t)\} = \int_0^\infty e^{at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{(a-s)t}\,dt = \int_0^\infty f(t)e^{-(s-a)t}\,dt = H(s-a).$$

Example: Find $L\{te^{2t}\}$ and $L\{t^2e^{2t}\}$.

Solution: Start with $L\{t\} = \frac{1}{s^2}$ and $L\{t^2\} = \frac{2}{s^3}$.

The presence of e^{2t} suggest to shift both results by 2 units. Thus,

$$L\{te^{2t}\} = \frac{1}{(s-2)^2} \quad \text{and} \quad L\{t^2e^{2t}\} = \frac{2}{(s-2)^3}$$

This is a repeat of the Example from Slide 4. Both methods work.

Example: Find $L\{e^{-3t}\cos(5t)\}$.

Solution: We have $L\{\cos(5t)\} = \frac{s}{s^2+25}$. Now, shift the result by -3 units:

$$L\{e^{-3t}\cos(5t)\} = \frac{s-(-3)}{\left(s-(-3)\right)^2+25} = \frac{s+3}{(s+3)^2+25}.$$

Example: Find $L\{te^{3t}\cos(5t)\}$.

Solution: Start with $H(s) = L\{\cos(5t)\} = \frac{s}{s^2+25}$. Differentiate H, so we get $H'(s) = \frac{25-s^2}{(s^2+25)^2}$.

Since $L\{-tf(t)\} = H'(s)$, we have $L\{t\cos(t)\} = \frac{s^2-25}{(s^2+25)^2}$. (The negative was moved to the right side and distributed into the numerator).

Then, the e^{3t} suggests to shift the result 3 units: $L\{te^{3t}\cos(5t)\} = \frac{(s-3)^2-25}{((s-3)^2+25)^2} = \frac{s^2-6s-16}{(s^2-6s+34)^2}$.

- **The Gamma Function:** If $y = t^n$, where $n > -1$, then $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, where Γ represents the gamma function.

Proof: Using the Laplace Transform, we have $L\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$. Using integration by parts where $u = t^n$ and $dv = e^{-st}$, we have $du = nt^{n-1}$ and $v = -\frac{1}{s}e^{-st}$. Thus,

$$L\{t^n\} = \int_0^\infty t^n e^{-st}\,dt = \left[-\frac{1}{s}t^n e^{-st}\right]_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}(nt^{n-1})\,dt\,.$$

The term $\left[-\frac{1}{s}t^n e^{-st}\right]_0^\infty = 0$ after evaluation.

Simplified, we get $\int_0^\infty t^n e^{-st}\,dt = \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt$. This is the same as $L\{t^n\} = \frac{n}{s}L\{t^{n-1}\}$. This can be extended, e.g. $L\{t^n\} = \frac{n}{s}L\{t^{n-1}\} = \frac{n}{s}\left(\frac{n-1}{s}L\{t^{n-2}\}\right)$, or $L\{t^n\} = \frac{n(n-1)}{s^2}L\{t^{n-2}\}$, and so on.

From the last slide, we showed that $L\{t^n\} = \frac{n}{s}L\{t^{n-1}\} = \frac{n(n-1)}{s^2}L\{t^{n-2}\}$ and so on.

If n is an integer such that $n \geq 0$, then this formula is the same as $L\{t^n\} = \frac{n!}{s^{n+1}}$ (Try it for $n = 3$, for example).

If n is not an integer such that $n > -1$, then the numerator is given by the gamma function $\Gamma(n+1)$, where $\Gamma(n+1) = \int_0^\infty t^n e^{-t}\,dt$. The gamma function has the property that when n is a non-negative integer, then $\Gamma(n+1) = n!$, but it "extends" the notion of factorial to include non-integers greater that –1. The integral $\int_0^\infty t^n e^{-t}\,dt$ is usually evaluated using numerical methods (e.g. a calculator).

Using the relationship $\Gamma(n+1) = n!$, then $\Gamma\left(\frac{3}{2}\right) = \left(\frac{1}{2}\right)!$, which is found by evaluating $\int_0^\infty t^{1/2}e^{-t}\,dt$. A calculator shows that this is about 0.88623. Using substitutions and integrating in polar coordinates, it can be shown that $\left(\frac{1}{2}\right)! = \int_0^\infty t^{1/2}e^{-t}\,dt = \frac{1}{2}\sqrt{\pi}$. Try it on your calculator!

Example: Find $L\{\sqrt{t}\}$.

Solution: We have $L\{\sqrt{t}\} = L\{t^{1/2}\} = \frac{\Gamma(3/2)}{s^{1/2+1}} = \frac{\Gamma(3/2)}{s^{3/2}} \approx \frac{0.88623}{s^{3/2}}$.

- **The $f(ct)$ Rule:** If $H(s) = L\{f(t)\}$, then $L\{f(ct)\} = \frac{1}{c}H\left(\frac{s}{c}\right)$.

Proof: Start with $L\{f(ct)\} = \int_0^\infty f(ct)e^{-st}\,dt$. Let $u = ct$ so that $t = \frac{u}{c}$ and $dt = \frac{du}{c}$. Make the substitutions:

$$L\{f(ct)\} = \int_0^\infty f(ct)e^{-st}\,dt = \int_0^\infty f(u)e^{-s\left(\frac{u}{c}\right)}\frac{du}{c} = \frac{1}{c}\int_0^\infty f(u)e^{\left(-\frac{s}{c}\right)u}\,du = \frac{1}{c}H\left(\frac{s}{c}\right).$$

Example: Find $L\{\sin(2t)\}$ using the above property.

We already know that $L\{\sin(bt)\} = b/(s^2 + b^2)$, so we should expect that we get $L\{\sin(2t)\} = 2/(s^2 + 4)$.

Solution: Starting with $L\{\sin(t)\} = \frac{1}{s^2+1}$ and observing that $c = 2$, we have

$$L\{\sin(2t)\} = \frac{1}{2}\cdot\frac{1}{\left(\frac{s}{2}\right)^2 + 1} = \frac{1}{2}\cdot\frac{1}{\left(\frac{s^2+4}{4}\right)} = \frac{1}{2}\cdot\left(\frac{4}{s^2+4}\right) = \frac{2}{s^2+4}.$$

Hey... it worked! :)