Solving IVPs using Laplace Transforms

MAT 275

Advantages of using Laplace Transforms to Solve IVPs

- It converts an IVP into an algebraic process in which the solution of the equation is the solution of the IVP.

- It handles initial conditions up front, not at the end of the process.

- It is “algorithmic” in that it follows a set process.

- It handles non-homogeneous forcing functions, and is especially useful when the forcing function is discontinuous or an impulse.

- It is cool.

Example: Solve $y'' + 2y' - 15y = 2t, y(0) = 1, y'(0) = 0$.

Solution: Apply the Laplace Transform operator to both sides:

$$L\{y'' + 2y' - 15y\} = L\{2t\}.$$

By linearity, distribute the operator and move coefficients to the front:

$$L\{y''\} + 2L\{y'\} - 15L\{y\} = 2L\{t\}.$$

Expand the left side:

$$\underbrace{s^2L\{y\} - sy(0) - y'(0)}_{L\{y''\}} + 2\underbrace{\left(sL\{y\} - y(0)\right)}_{L\{y'\}} - 15L\{y\} = 2L\{t\}.$$

Note that this is the step in which the initial conditions are handled. (continued…)

From the previous slide, we have

$$s^2L\{y\} - sy(0) - y'(0) + 2\big(sL\{y\} - y(0)\big) - 15L\{y\} = 2L\{t\}.$$

Since $y(0) = 1$ and $y'(0) = 0$, we have

$$s^2L\{y\} - s \cdot 1 - 0 + 2(sL\{y\} - 1) - 15L\{y\} = 2L\{t\}$$

Simplify a little, distributing to clear parentheses:

$$s^2L\{y\} - s + 2sL\{y\} - 2 - 15L\{y\} = 2L\{t\}$$

On the right side, we have $L\{t\} = \frac{1}{s^2}$, so we have

$$s^2L\{y\} - s + 2sL\{y\} - 2 - 15L\{y\} = \frac{2}{s^2}\ .$$

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From the last slide, we have $s^2L\{y\} - s + 2sL\{y\} - 2 - 15L\{y\} = \frac{2}{s^2}$.

Now we start to isolate $L\{y\}$:

$$L\{y\}(s^2 + 2s - 15) = \frac{2}{s^2} + s + 2.$$

Get a common denominator on the right side:

$$L\{y\}(s^2 + 2s - 15) = \frac{s^3 + 2s^2 + 2}{s^2} .$$

Divide, and we have isolated $L\{y\}$:

$$L\{y\} = \frac{s^3 + 2s^2 + 2}{s^2(s^2 + 2s - 15)} .$$

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From the last slide, we have $L\{y\} = \frac{s^3+2s^2+2}{s^2(s^2+2s-15)}$.

The solution to the differential equation is found by inverting the right side. In other words,

$$y = L^{-1}\left\{\frac{s^3+2s^2+2}{s^2(s^2+2s-15)}\right\} .$$

We need to “break apart” the big expression into smaller summands. The denominator factors:

$$\frac{s^3+2s^2+2}{s^2(s^2+2s-15)} = \frac{s^3+2s^2+2}{s^2(s+5)(s-3)} .$$

We’ll use partial fractions, starting next slide…

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We need to decompose $\frac{s^3+2s^2+2}{s^2(s+5)(s-3)}$ using partial fractions.

The factor s^2 is a linear factor of multiplicity 2 so it results in two fractional summands, while the factors $(s+5)$ and $(s-3)$ are each multiplicity 1, so they result in one summand each:

$$\frac{s^3+2s^2+2}{s^2(s+5)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+5} + \frac{D}{s-3}\ .$$

Now get a common denominator and recompose:

$$\frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+5} + \frac{D}{s-3} = \frac{As(s+5)(s-3) + B(s+5)(s-3) + Cs^2(s-3) + Ds^2(s+5)}{s^2(s+5)(s-3)}$$

Then equate numerators:

$$s^3+2s^2+2 = As(s+5)(s-3) + B(s+5)(s-3) + Cs^2(s-3) + Ds^2(s+5)$$

From the last slide, we have

$$s^3 + 2s^2 + 2 = As(s+5)(s-3) + B(s+5)(s-3) + Cs^2(s-3) + Ds^2(s+5)$$

On the right side, multiply to clear parentheses:

$$s^3 + 2s^2 + 2 = As^3 + 2As^2 - 15As + Bs^2 + 2Bs - 15B + Cs^3 - 3Cs^2 + Ds^3 + 5Ds^2$$

Now collect terms on the right side according to powers of s:

$$s^3 + 2s^2 + 2 = s^3[A + C + D] + s^2[2A + B - 3C + 5D] + s[-15A + 2B] - 15B.$$

The coefficients on the left must match the coefficients on the right. This means

For s^3: $A + C + D = 1,$ For s: $-15A + 2B = 0,$
For s^2: $2A + B - 3C + 5D = 2,$ For the constants: $-15B = 2.$

The four equations from the last slide form a system:

$$\begin{gathered} A + C + D = 1 \\ 2A + B - 3C + 5D = 2 \\ -15A + 2B = 0 \\ -15B = 2 \end{gathered}$$

Zero-filling, we have

$$\begin{gathered} A + 0B + C + D = 1 \\ 2A + B - 3C + 5D = 2 \\ -15A + 2B + 0C + 0D = 0 \\ 0A - 15B + 0C + 0D = 2 \end{gathered}$$

Enter this matrix in your calculator:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & -3 & 5 & 2 \\ -15 & 2 & 0 & 0 & 0 \\ 0 & -15 & 0 & 0 & 2 \end{bmatrix}$$

Using the RREF feature on a calculator, we find that

$$A = -\frac{4}{225}, \qquad B = -\frac{2}{15}, \qquad C = \frac{73}{200}, \qquad D = \frac{47}{72}.$$

Believe it or not, we’re almost done!

So we now have $\frac{s^3+2s^2+2}{s^2(s+5)(s-3)} = \frac{-4/225}{s} + \frac{-2/15}{s^2} + \frac{73/200}{s+5} + \frac{47/72}{s-3}$.

Thus, the solution of $y'' + 2y' - 15y = 2t, y(0) = 1, y'(0) = 0$ is

$$y = L^{-1}\left\{\frac{s^3 + 2s^2 + 2}{s^2(s+5)(s-3)}\right\} = -\frac{4}{225}L^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{15}L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{73}{200}L^{-1}\left\{\frac{1}{s+5}\right\} + \frac{47}{72}L^{-1}\left\{\frac{1}{s-3}\right\}.$$

Recall that $L^{-1}\left\{\frac{1}{s}\right\} = 1$, $L^{-1}\left\{\frac{1}{s^2}\right\} = t$, $L^{-1}\left\{\frac{1}{s+5}\right\} = e^{-5t}$ and $L^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$. The coefficients we just found simply move outside.

And finally… the solution is:

$$y = -\frac{4}{225} - \frac{2}{15}t + \frac{73}{200}e^{-5t} + \frac{47}{72}e^{3t}.$$

(In the previous example, every step was shown. In this example, some steps will be combined)

Example: Solve the IVP $y'' + 2y' + 10y = t^2,\ \ y(0) = 1, y'(0) = -2.$

Solution: Apply the Laplace Transform operator to both sides and simplify:

$$L\{y'' + 2y' + 10y\} = L\{t^2\}$$

Distribute

$$L\{y''\} + 2L\{y'\} + 10L\{y\} = L\{t^2\}$$

Use forms for $L\{y''\}$ and $L\{y\}$

$$s^2L\{y\} - sy(0) - y'(0) + 2\big(sL\{y\} - y(0)\big) + 10L\{y\} = \frac{2}{s^3}$$

Apply initial conditions

$$s^2L\{y\} - s + 2 + 2sL\{y\} - 2 + 10L\{y\} = \frac{2}{s^3}$$

$$L\{y\}(s^2 + 2s + 10) = \frac{s^4 + 2}{s^3} \quad \text{so that} \quad L\{y\} = \frac{s^4 + 2}{s^3(s^2 + 2s + 10)}.$$

Collect terms, get a common denominator on the right.

From the previous slide, we have $L\{y\} = \frac{s^4+2}{s^3(s^2+2s+10)}$.

We need to find $y = L^{-1}\left\{\frac{s^4+2}{s^3(s^2+2s+10)}\right\}$. Using partial fractions, the expression s^3 is a linear factor s with multiplicity 3, so it results in three partial fraction summands. The expression $s^2 + 2s + 10$ is an irreducible quadratic. Thus, the partial fraction decomposition is

$$\frac{s^4 + 2}{s^3(s^2 + 2s + 10)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 2s + 10} .$$

Now, recompose by getting a common denominator on the right side:

$$\frac{As^2(s^2 + 2s + 10) + Bs(s^2 + 2s + 10) + C(s^2 + 2s + 10) + (Ds + E)s^3}{s^3(s^2 + 2s + 10)}$$

Now, equate numerators:

$$s^4 + 2 = As^2(s^2 + 2s + 10) + Bs(s^2 + 2s + 10) + C(s^2 + 2s + 10) + (Ds + E)s^3$$

Distribute to clear parentheses

$$s^4 + 2 = As^4 + 2As^3 + 10As^2 + Bs^3 + 2Bs^2 + 10Bs + Cs^2 + 2Cs + 10C + Ds^4 + Es^3$$

Collect terms by powers of *s*.

$$s^4 + 2 = s^4[A + D] + s^3[2A + B + E] + s^2[10A + 2B + C] + s[10B + 2C] + [10C].$$

Thus, we have five equations in five variables:

$$A + D = 1$$

The coefficient of s^4 is 1

$$2A + B + E = 0$$

$$10A + 2B + C = 0$$

The coefficients of s^3, s^2 and s are 0

$$10B + 2C = 0$$

$$10C = 2.$$

The constant is 2

From the previous slide, we have five equations:

$$A + D = 1, \quad 2A + B + E = 0, \quad 10A + 2B + C = 0, \quad 10B + 2C = 0, \quad 10C = 2.$$

The last equation gives $C = \frac{1}{5}$. This is used in the fourth equation, getting $B = -\frac{1}{25}$. These two values are substituted in the third equation, giving $A = -\frac{3}{250}$.

The value for A is substituted into the top equation, so that $D = \frac{253}{250}$. Finally, A and B are substituted in the second equation, giving $E = \frac{8}{125}$.

Thus, the partial fraction decomposition is

$$\frac{s^4 + 2}{s^3(s^2 + 2s + 10)} = \frac{-\frac{3}{250}}{s} + \frac{-\frac{1}{25}}{s^2} + \frac{\frac{1}{5}}{s^3} + \frac{\left(\frac{253}{250}\right)s + \frac{8}{125}}{s^2 + 2s + 10} .$$

We have $\frac{s^4+2}{s^3(s^2+2s+10)} = \frac{-\frac{3}{250}}{s} + \frac{-\frac{1}{25}}{s^2} + \frac{\frac{1}{5}}{s^3} + \frac{\left(\frac{253}{250}\right)s+\frac{8}{125}}{s^2+2s+10}$.

We'll concentrate on the last term for now. Complete the square:

$$\frac{\left(\frac{253}{250}\right)s + \frac{8}{125}}{s^2 + 2s + 10} = \frac{\left(\frac{253}{250}\right)s + \frac{8}{125}}{(s+1)^2 + 9} .$$

We intend to use the Laplace Transforms $L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$ and $L\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$. Thus, we "need" an $(s+1)$ in the numerator:

$$\frac{\frac{253}{250}s + \frac{8}{125}}{(s+1)^2 + 9} = \frac{\frac{253}{250}(s+1-1) + \frac{8}{125}}{(s+1)^2 + 9} = \frac{\frac{253}{250}(s+1) - \frac{253}{250} + \frac{8}{125}}{(s+1)^2 + 9} = \frac{\frac{253}{250}(s+1) - \frac{237}{250}}{(s+1)^2 + 9}$$

We have $\frac{s^4+2}{s^3(s^2+2s+10)} = \frac{-\frac{3}{250}}{s} + \frac{-\frac{1}{25}}{s^2} + \frac{\frac{1}{5}}{s^3} + \frac{\left(\frac{253}{250}\right)s+\frac{8}{125}}{s^2+2s+10}$.

Now we have

$$\frac{s^4+2}{s^3(s^2+2s+10)} = \frac{-\frac{3}{250}}{s} + \frac{-\frac{1}{25}}{s^2} + \frac{\frac{1}{5}}{s^3} + \frac{\frac{253}{250}(s+1) - \frac{237}{250}}{(s+1)^2+9}$$

Split at this negative sign

$$\frac{-\frac{3}{250}}{s} + \frac{-\frac{1}{25}}{s^2} + \frac{\frac{1}{5}}{s^3} + \frac{\frac{253}{250}(s+1)}{(s+1)^2+9} - \frac{\frac{237}{250}}{(s+1)^2+9} .$$

Thus, $y = L^{-1}\left\{\frac{-\frac{3}{250}}{s} + \frac{-\frac{1}{25}}{s^2} + \frac{\frac{1}{5}}{s^3} + \frac{\frac{253}{250}(s+1)}{(s+1)^2+9} - \frac{\frac{237}{250}}{(s+1)^2+9}\right\}$.

The final answer is on the next side.

We have $y = L^{-1}\left\{\frac{-\frac{3}{250}}{s} + \frac{-\frac{1}{25}}{s^2} + \frac{\frac{1}{5}}{s^3} + \frac{\frac{253}{250}(s+1)}{(s+1)^2+9} - \frac{\frac{237}{250}}{(s+1)^2+9}\right\}$.

Balance with constants when necessary:

$$y = -\frac{3}{250}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{25}L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{5}\cdot\frac{1}{2}L^{-1}\left\{\frac{2}{s^3}\right\} + \frac{253}{250}L^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} - \frac{237}{250}\cdot\frac{1}{3}L^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

This fraction simplifies to 79/250

$$y = -\frac{3}{250} - \frac{1}{25}t + \frac{1}{10}t^2 + \frac{253}{250}e^{-t}\cos(3t) - \frac{79}{250}e^{-t}\sin(3t).$$