Using Laplace Transforms to Solve IVPs with Discontinuous Forcing Functions

MAT 275

Example: Find the solution of the IVP

$$y'' + 2y' + 5y = \begin{cases} 0, & t < 4 \\ 1, & t \geq 4 \end{cases}, \qquad y(0) = 1, y'(0) = -1.$$

Solution: Rewrite the forcing function using the $u_c(t)$ notation:

$$y'' + 2y' + 5y = u_4(t), \qquad y(0) = 1, y'(0) = -1.$$

Now apply the Laplace Transform Operator to both sides and simplify:

Recall that the operator is linear, so distribute and move coefficients outside.

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{u_4(t)\}$$

$$s^2L\{y\} - sy(0) - y'(0) + 2\big(sL\{y\} - y(0)\big) + 5L\{y\} = L\{u_4(t)\}$$

1   -1   1

$$s^2L\{y\} - s + 1 + 2sL\{y\} - 2 + 5L\{y\} = \frac{e^{-4s}}{s}$$

Recall that $L\{u_c(t)\} = \frac{e^{-cs}}{s}$

Distribute the 2.

Collect terms

$$L\{y\}[s^2 + 2s + 5] = \frac{e^{-4s}}{s} + s + 1.$$

Now isolate $L\{y\}$:

$$L\{y\} = \frac{e^{-4s}}{s(s^2+2s+5)} + \frac{s+1}{s^2+2s+5}.$$

Leave the e^{-4s} in one term, and all else combined into another term.

The solution is the inversion of the above expressions:

$$y = L^{-1}\left\{\frac{e^{-4s}}{s(s^2+2s+5)} + \frac{s+1}{s^2+2s+5}\right\}.$$

We'll work on the term without the e^{-4s} first. Note that the denominator s^2+2s+5 is an irreducible quadratic over the reals, so we complete the square:

$$\frac{s+1}{s^2+2s+5} = \frac{s+1}{(s+1)^2+4}.$$

$a = -1$, $b = 2$

This form exactly fits $L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$. Thus, $L^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} = e^{-t}\cos(2t)$.

From the last slide, we have $y = L^{-1}\left\{\frac{e^{-4s}}{s(s^2+2s+5)} + \frac{s+1}{s^2+2s+5}\right\}$.

Now we'll work on finding $L^{-1}\left\{\frac{e^{-4s}}{s(s^2+2s+5)}\right\}$. The e^{-4s} will result in $u_4(t)$ appearing in the final result. So we mentally note this fact, then "ignore" it for the next few steps, as we rewrite $\frac{1}{s(s^2+2s+5)}$ into smaller fractions using partial fraction decomposition:

$$\frac{1}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5} = \frac{A(s^2+2s+5)+(Bs+C)s}{s(s^2+2s+5)}.$$

Equating the numerators, we have $1 = A(s^2+2s+5) + (Bs+C)s$.

Collecting terms according to powers of s, we have $1 = \underbrace{(A+B)}_{=0}s^2 + \underbrace{(2A+C)}_{=0}s + \underbrace{5A}_{=1}$.

Thus, $A = \frac{1}{5}$, $B = -\frac{1}{5}$ and $C = -\frac{2}{5}$.

So now we have

Since B and C were both negative, the negative was factored to the front.

$$\frac{1}{s(s^2+2s+5)} = \frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}s+\frac{2}{5}}{s^2+2s+5}.$$

Completing the square on the second term, we have $s^2+2s+5=(s+1)^2+4$. Thus, we need to have $s+1$ in the numerator:

Distribute the leading negative sign.

Multiply inside by 2 and outside by 1/2 to prepare this for an inversion.

$$\frac{\frac{1}{5}s+\frac{2}{5}}{s^2+2s+5} = \frac{\frac{1}{5}(s+1-1)+\frac{2}{5}}{(s+1)^2+4} = \frac{\frac{1}{5}(s+1)+\frac{1}{5}}{(s+1)^2+4} = \frac{\frac{1}{5}(s+1)}{(s+1)^2+4} + \frac{1}{2}\cdot\frac{\frac{1}{5}(2)}{(s+1)^2+4}.$$

Finally, we have that $L^{-1}\left\{\frac{e^{-4s}}{s(s^2+2s+5)}\right\} = L^{-1}\left\{e^{-4s}\left(\frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}(s+1)}{(s+1)^2+4} - \frac{1}{2}\cdot\frac{\frac{1}{5}(2)}{(s+1)^2+4}\right)\right\}.$

The whole solution is pieced together on the next slide.

We have $y = L^{-1}\left\{\frac{e^{-4s}}{s(s^2+2s+5)} + \frac{s+1}{s^2+2s+5}\right\}$.

From slide 3, we had $L^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} = e^{-t}\cos(2t)$.

From the last slide, we had $L^{-1}\left\{\frac{e^{-4s}}{s(s^2+2s+5)}\right\} = L^{-1}\left\{e^{-4s}\left(\frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}(s+1)}{(s+1)^2+4} - \frac{1}{2}\cdot\frac{\frac{1}{5}(2)}{(s+1)^2+4}\right)\right\}$.

Combine 1/2 and 1/5

Note that $L^{-1}\left\{\frac{\frac{1}{5}(s+1)}{(s+1)^2+4}\right\} = \frac{1}{5}e^{-t}\cos(2t)$ and $L^{-1}\left\{\frac{1}{10}\cdot\frac{2}{(s+1)^2+4}\right\} = \frac{1}{10}e^{-t}\sin(2t)$.

This gives $u_4(t)\left(\frac{1}{5} - \frac{1}{5}e^{-(t-4)}\cos\big(2(t-4)\big) - \frac{1}{10}e^{-(t-4)}\sin\big(2(t-4)\big)\right)$, where we must state the shift of 4 units to the right. The entire solution is

$$y = e^{-t}\cos(2t) + u_4(t)\left(\frac{1}{5} - \frac{1}{5}e^{-(t-4)}\cos\big(2(t-4)\big) - \frac{1}{10}e^{-(t-4)}\sin\big(2(t-4)\big)\right).$$

The solution of $y'' + 2y' + 5y = \begin{cases} 0, & t < 4 \\ 1, & t \geq 4 \end{cases}$, $y(0) = 1, y'(0) = -1$ is

$$y = e^{-t}\cos(2t) + u_4(t)\left(\frac{1}{5} - \frac{1}{5}e^{-(t-4)}\cos\big(2(t-4)\big) - \frac{1}{10}e^{-(t-4)}\sin\big(2(t-4)\big)\right).$$

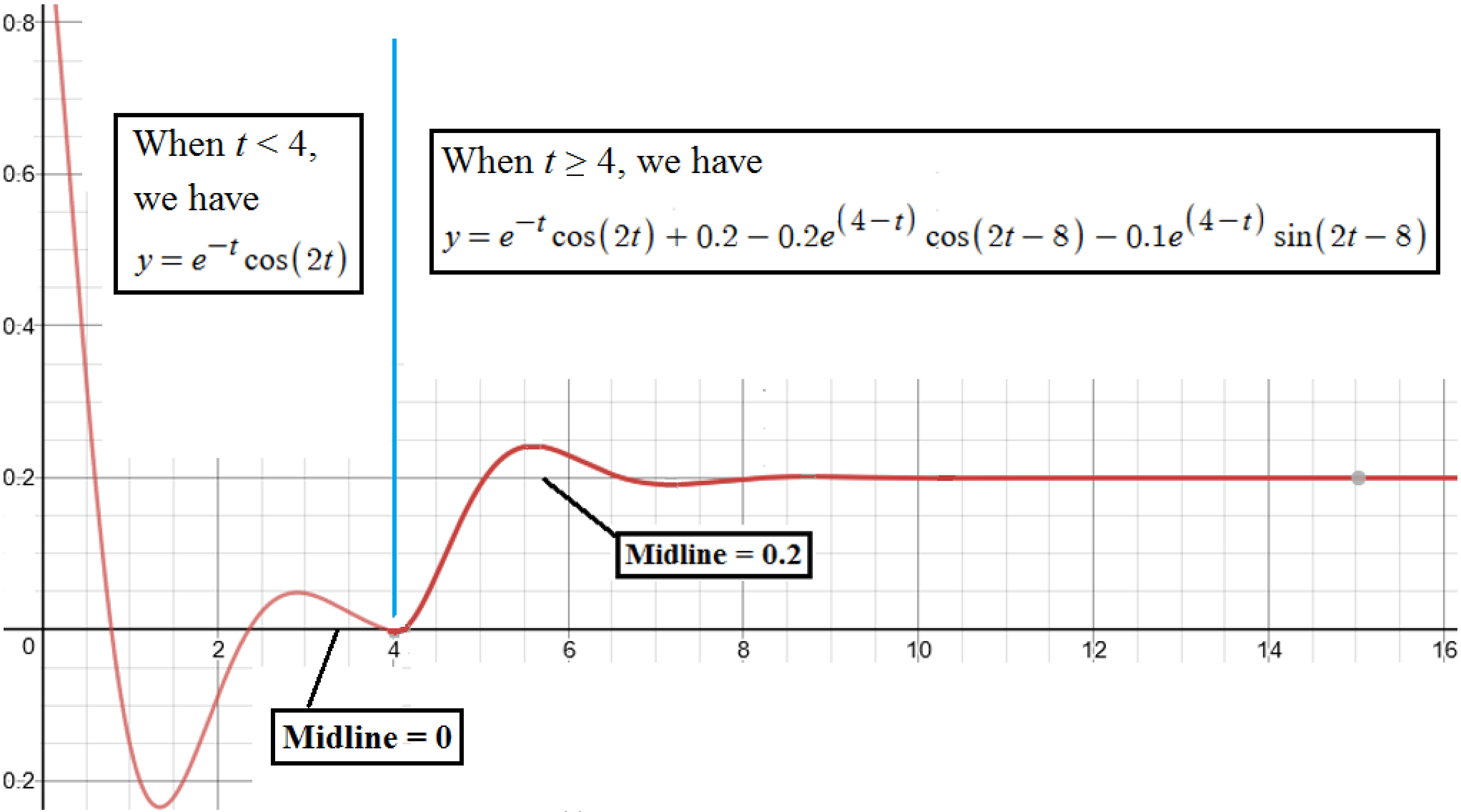
- When $t < 4$, then $u_4(t) = 0$ and we have $y = e^{-t}\cos(2t)$.
- When $t \geq 4$, then $u_4(t) = 1$ and we have

$$y = e^{-t}\cos(2t) + \frac{1}{5} - \frac{1}{5}e^{-(t-4)}\cos\big(2(t-4)\big) - \frac{1}{10}e^{-(t-4)}\sin\big(2(t-4)\big).$$

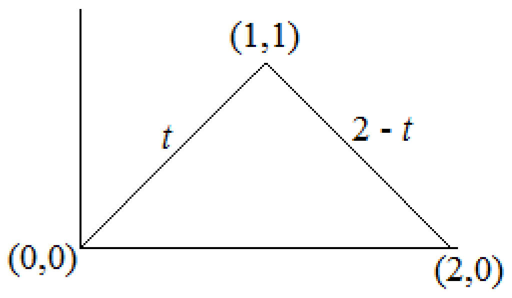
- At $t = 4$, the function is continuous.

The graph is on the next slide.

Graph of: $$y = e^{-t}\cos(2t) + u_4(t)\left(\frac{1}{5} - \frac{1}{5}e^{-(t-4)}\cos\big(2(t-4)\big) - \frac{1}{10}e^{-(t-4)}\sin\big(2(t-4)\big)\right)$$



Example: Solve $y'' + 9y = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \end{cases}, \; y(0) = 0, y'(0) = 0.$



This is what the forcing function looks like.

Solution: We need to write the forcing function using $u_c(t)$ notation.

- When $0 \le t < 1$, we have t, which does not need a leading "u" for now.
- When $1 \le t < 2$, we need to "turn off" t and "turn on" $2 - t$. Thus we have:

$$t - u_1(t)t + u_1(t)(2 - t)$$

$$t - u_1(t)t + 2u_1(t) - u_1(t)t \quad \text{Distribute}$$

$$t + 2u_1(t) - 2u_1(t)t \quad \text{Collect terms}$$

$$t + u_1(t)(2 - 2t). \quad \text{Factor out the } u_1(t)$$

Note that when $t = 1$, then $u_1(t) = 1$, so that the last line is $t + 1 \cdot (2 - 2t) = t + 2 - 2t$, which simplifies to $2 - t$, just like in the original statement.

The IVP is now written $y'' + 9y = t + u_1(t)(2 - 2t), \quad y(0) = 0, y'(0) = 0.$

We have $y'' + 9y = t + u_1(t)(2 - 2t), \quad y(0) = 0, y'(0) = 0.$

Apply the Laplace Transform operator to both sides:

$$L\{y''\} + 9L\{y\} = L\{t + u_1(t)(2 - 2t)\}$$

$$s^2L\{y\} - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + 9L\{y\} = L\{t\} + L\{u_1(t)2 - u_1(t)(2(t - 1 + 1))\}$$

Distribute the u term. Build in the shift

$$s^2L\{y\} + 9L\{y\} = L\{t\} + L\{u_1(t)2 - u_1(t)2(t - 1) - u_1(t)2\}$$ Distribute the u through

$$L\{y\}[s^2 + 9] = L\{t\} - L\{u_1(t)2(t - 1)\}$$ The $L\{u_1(t)2\}$ cancel

$$L\{y\}[s^2 + 9] = \frac{1}{s^2} - \frac{2e^{-s}}{s^2}$$

$$L\{y\} = \frac{1}{s^2(s^2 + 9)} - \frac{2e^{-s}}{s^2(s^2 + 9)}.$$ Isolate $L\{y\}$

We now invert $L\{y\}=\frac{1}{s^2(s^2+9)}-\frac{2e^{-s}}{s^2(s^2+9)}$. For $L^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\}$, we use partial fractions to simplify the expression:

$$\frac{1}{s^2(s^2+9)}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+9}=\frac{As(s^2+9)+B(s^2+9)+(Cs+D)s^2}{s^2(s^2+9)}.$$

The numerator at upper right is written in terms of powers of s:

We'll use this same expression again for the other inversion to be performed.

$$As(s^2+9)+B(s^2+9)+(Cs+D)s^2=(A+C)s^3+(B+D)s^2+9As+9B.$$

Equating numerators, we have $\underbrace{(A+C)}_{=0}s^3+\underbrace{(B+D)}_{=0}s^2+\underbrace{9A}_{=0}s+\underbrace{9B}_{=1}=1.$

Thus, $B=\frac{1}{9}$, and since $B+D=0$, then $D=-\frac{1}{9}$. Since $9A=0$, then $A=0$, forcing $C=0$.

Remember, $L\{\sin(bt)\}=b/(s^2+b^2)$, so $b=3$, and we need a 3 on top and 1/3 outside.

We have, $y=L^{-1}\left\{\frac{1}{s^2(s^2+9)}\right\}=\frac{1}{9}L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{1}{9}\cdot\frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\}=\frac{1}{9}t-\frac{1}{27}\sin(3t),\quad 0\le t<1.$

Now we find $L^{-1}\left\{\frac{2e^{-s}}{s^2(s^2+9)}\right\}$. The expression $\frac{2}{s^2(s^2+9)}$ decomposes as

$$\frac{2}{s^2(s^2+9)} = \frac{2}{9}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right).$$

Same as last slide. B = 2/9, D = -2/9

The e^{-s} now inverts to $u_1(t)$.

Thus, $y = L^{-1}\left\{\frac{2e^{-s}}{s^2(s^2+9)}\right\} = u_1(t)\cdot\frac{2}{9}\cdot\left[L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3}\cdot L^{-1}\left\{\frac{3}{s^2+9}\right\}\right]$.

The shifts are written back in

This gives $y = \frac{2}{9}u_1(t)\left((t-1) - \frac{1}{3}\sin\big(3(t-1)\big)\right)$, where $1 \le t < 2$.

The solution of $y'' + 9y = t + u_1(t)(2-2t),\quad y(0) = 0, y'(0) = 0$ is

$$y = \frac{1}{9}t - \frac{1}{27}\sin(3t) + \frac{2}{9}u_1(t)\left(t - 1 - \frac{1}{3}\sin(3t-3)\right)$$

Some simplification took place.

The graph of $y = \frac{1}{9}t - \frac{1}{27}\sin(3t) + \frac{2}{9}u_1(t)\left(t - 1 - \frac{1}{3}\sin(3t - 3)\right)$ is