Laplace Transforms

MAT 275

Let $y = f(t)$ be a function. Its **Laplace Transform**, written $H(s) = L\{y(t)\}$, is a function in variable s, defined by

$$H(s) = L\{y(t)\} = \int_0^\infty f(t)e^{-st}\,dt\,.$$

Case 1 (Constants). Let $f(t) = c$, where c is any constant. Then

$$H(s) = L\{c\} = \int_0^\infty ce^{-st}\,dt = c\int_0^\infty e^{-st}\,dt\,.$$

The integral $\int_0^\infty e^{-st}\,dt$ is found using limits:

Equals 0 as *b* trends to infinity

= 1

$$\lim_{b\to\infty}\int_0^b e^{-st}\,dt = \lim_{b\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^b = \lim_{b\to\infty}\left[-\frac{1}{s}\left(e^{-sb} - (e^0)\right)\right] = \frac{1}{s}.$$

$$\boxed{\text{Thus, when } f(t) = c, \text{ then } H(s) = L\{c\} = \frac{c}{s}\,.}$$

Case 2 (exponential, base-*e*): Let $f(t) = e^{at}$. Then

$$H(s) = L\{e^{at}\} = \int_0^{\infty} e^{at} e^{-st}\, dt = \int_0^{\infty} e^{(a-s)t}\, dt\,.$$

The integral is evaluated using limits:

$$\lim_{b\to\infty} \int_0^b e^{(a-s)t}\, dt = \lim_{b\to\infty} \left[\frac{1}{a-s} e^{(a-s)t}\right]_0^b = \lim_{b\to\infty} \left[\frac{1}{a-s}\left(e^{(a-s)b} - (e^0)\right)\right].$$

When $s > a$, then $\lim_{b\to\infty}\left[e^{(a-s)b}\right] = 0$ since the coefficient of b is negative. Also, $e^0 = 1$. So we're left with $\frac{1}{a-s}(0-1) = \frac{1}{s-a}$.

$$\boxed{\text{Thus, when } f(t) = e^{at}, \text{ we have } H(s) = L\{e^{at}\} = \frac{1}{s-a}\,.}$$

Laplace Transforms for other types of function

Usually, we do not bother to calculate Laplace Transforms by hand for most functions. Instead, we make up a table for the most common functions. For example, if $y = t^n$, where n is a positive integer, then

$$H(s) = L\{t^n\} = \frac{n!}{s^{n+1}}\ .$$

If $y = \sin(bt)$, then

$$H(s) = L\{\sin(bt)\} = \frac{b}{s^2 + b^2}\ .$$

All of these were found by integration by parts.

If $y = \cos(bt)$, then

$$H(s) = L\{\cos(bt)\} = \frac{s}{s^2 + b^2}\ .$$

List of Common Laplace Transforms

Function	Laplace Transform
$y = c$ (any constant)	$H(s) = \frac{c}{s}$
$y = t^n$	$H(s) = \frac{n!}{s^{n+1}}$
$y = e^{at}$	$H(s) = \frac{1}{s-a}$
$y = \cos bt$	$H(s) = \frac{s}{s^2+b^2}$
$y = \sin bt$	$H(s) = \frac{b}{s^2+b^2}$
$y = e^{at}\cos bt$	$H(s) = \frac{s-a}{(s-a)^2+b^2}$
$y = e^{at}\sin bt$	$H(s) = \frac{b}{(s-a)^2+b^2}$

Memorize this table!

Laplace Transforms of Derivatives

Let $y = f(t)$ and suppose initial conditions $y = f(0)$ and $y' = f'(0)$ are given. Then it is possible to find the Laplace Transform of a derivative. These are two that are most common:

For $y' = f'(t)$, we have $H(s) = L\{y'\} = sL\{y\} - y(0)$.

For $y'' = f''(t)$, we have $H(s) = L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$.

The notation $y(0)$ is the same as $y = f(0)$ and $y'(0)$ is the same as $y' = f'(0)$.

Linearity of the Laplace Transform

The operator can be distributed and any coefficients move to the front.

$$L\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 L\{f_1(t)\} + c_2 L\{f_2(t)\}.$$

Example: Find $L\{t^4\}$.

Solution: Using the form $L\{t^n\} = \frac{n!}{s^{n+1}}$, we have $L\{t^4\} = \frac{4!}{s^{4+1}} = \frac{24}{s^5}$.

Example: Find $L\{e^{5t}\}$.

With practice, you'll get to know these forms better.

Solution: Using the form $L\{e^{at}\} = \frac{1}{s-a}$, we have $L\{e^{5t}\} = \frac{1}{s-5}$.

Example: Find $L\{\sin(7t)\}$.

Solution: Using the form $L\{\sin(bt)\} = \frac{b}{s^2+b^2}$, we have $L\{\sin(7t)\} = \frac{7}{s^2+49}$.

Example: Find $L\{e^{-2t}\cos(3t)\}$.

Solution: Using the form $L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$, we have $L\{e^{-2t}\cos(3t)\} = \frac{s+2}{(s+2)^2+9}$.

Use algebra when appropriate…

Example: Find $L\{(t+3)^2\}$.

Solution: Multiply the binomial: $L\{(t+3)^2\} = L\{t^2+6t+9\}$. Now, using the linearity of the operator, we have

$$L\{(t+3)^2\} = L\{t^2+6t+9\}$$

$$= L\{t^2\} + 6L\{t\} + L\{9\}$$

$$= \frac{2!}{s^{2+1}} + 6\left(\frac{1!}{s^{1+1}}\right) + 9\left(\frac{1}{s}\right)$$

$$= \frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\ .$$

Use trigonometric identities also…

Example: Find $L\{\sin^2(3t)\}$.

Solution: We use the identity $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$. Thus,

$$L\{\sin^2(3t)\} = L\left\{\frac{1}{2} - \frac{1}{2}\cos(6t)\right\}$$

$$= L\left\{\frac{1}{2}\right\} - L\left\{\frac{1}{2}\cos(6t)\right\}$$

$$= \frac{1}{2s} - \frac{1}{2}\left(\frac{s}{s^2+36}\right).$$

Inverting the Laplace Transform

As part of the solution process, we will need to invert the Laplace Transform, to find the function $f(t)$ such that

$$L^{-1}\{H(s)\} = f(t).$$

Some are easy to do by "inspection":

Example: Find $L^{-1}\left\{\frac{7}{s}\right\}$.

Solution: Since we know that $L\{c\} = \frac{c}{s}$, then it follows that $L^{-1}\left\{\frac{7}{s}\right\} = 7$.

Example: Find $L^{-1}\left\{\frac{s}{s^2+25}\right\}$.

Solution: Sine we know that $L\{\cos(bt)\} = \frac{s}{s^2+b^2}$, then it follows that $L^{-1}\left\{\frac{s}{s^2+25}\right\} = \cos(5t)$.

Inversions, continued…

Some cases, we need to "balance" the expression with constants and their reciprocals.

Example: Find $L^{-1}\left\{\frac{1}{s^6}\right\}$.

Solution: We know that $L\{t^n\} = \frac{n!}{s^{n+1}}$, so we conclude that n = 5 in this example. However, we need 5! = 120 in the numerator to fully agree with the form. So we multiply inside by 120, and outside by $\frac{1}{120}$, and we have

$$L^{-1}\left\{\frac{1}{s^6}\right\} = \frac{1}{120}L^{-1}\left\{\frac{120}{s^6}\right\} = \frac{1}{120}t^5.$$

Example: Find $L^{-1}\left\{\frac{3}{s^2+21}\right\}$.

Solution: This looks closest to the form $L\{\sin(bt)\} = \frac{b}{s^2+b^2}$. The 3 in the numerator can be moved to the front, and from the denominator, we infer that since $b^2 = 21$, we must have $b = \sqrt{21}$.

However, the form "needs" the b value in the numerator. So we multiply inside by $\sqrt{21}$ and outside by $\frac{1}{\sqrt{21}}$:

$$L^{-1}\left\{\frac{3}{s^2+21}\right\} = 3L^{-1}\left\{\frac{1}{s^2+21}\right\} = 3\left(\frac{1}{\sqrt{21}}\right)L^{-1}\left\{\frac{\sqrt{21}}{s^2+21}\right\} = \frac{3}{\sqrt{21}}\sin\left(\sqrt{21}t\right).$$

Rationalizing the denominator, an equivalent form is $L^{-1}\left\{\frac{3}{s^2+21}\right\} = \frac{\sqrt{21}}{7}\sin\left(\sqrt{21}t\right)$.

Example: Find $L^{-1}\left\{\frac{s}{s^2+4s+9}\right\}$.

Solution: The denominator does not factor over the Reals, so we complete the square:

$$\frac{s}{s^2+4s+9}=\frac{s}{(s+2)^2+5}\ .$$

This looks closest to $L\{e^{at}\cos(bt)\}=\frac{s-a}{(s-a)^2+b^2}$. We can conclude that $a=-2$ and $b=\sqrt{5}$.

But the numerator is not in the right form. We “need” $(s+2)$ in the numerator in order to agree with the form. So we add in 2 then subtract it back out, then split the numerator by grouping $s+2$ and -2 as separate terms:

$$\frac{s}{(s+2)^2+5}=\frac{s+2-2}{(s+2)^2+5}=\frac{s+2}{(s+2)^2+5}-\frac{2}{(s+2)^2+5}.$$

(continued…)

From the last screen, we have

$$\frac{s}{s^2+4s+9} = \frac{s}{(s+2)^2+5} = \frac{s+2-2}{(s+2)^2+5} = \frac{s+2}{(s+2)^2+5} - \frac{2}{(s+2)^2+5}\,.$$

The expression $\frac{s+2}{(s+2)^2+5}$ now <u>exactly</u> matches the form for $L\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}$.

The expression $\frac{2}{(s+2)^2+5}$ almost matches the form for $L\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}$. To make it match exactly, multiply inside by $\sqrt{5}$ and outside by $\frac{1}{\sqrt{5}}$, getting $\frac{2}{\sqrt{5}}\left(\frac{\sqrt{5}}{(s+2)^2+5}\right)$. Note that we moved the numerator 2 to the outside. Now everything matches. Thus,

$$L^{-1}\left\{\frac{s}{s^2+4s+9}\right\} = L^{-1}\left\{\frac{s+2}{(s+2)^2+5}\right\} - \frac{2}{\sqrt{5}}L^{-1}\left\{\frac{\sqrt{5}}{(s+2)^2+5}\right\} = e^{-2t}\cos\left(\sqrt{5}t\right) - \frac{2}{\sqrt{5}}e^{-2t}\sin\left(\sqrt{5}t\right).$$

Example: Find $L^{-1}\left\{\frac{1}{s(s^2+s-12)}\right\}$.

Solution: The denominator factors into linear factors:

$$\frac{1}{s(s^2+s-12)} = \frac{1}{s(s+4)(s-3)}.$$

We now use partial fraction decomposition:

$$\frac{1}{s(s+4)(s-3)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-3}.$$

Recomposing, we have

$$\frac{1}{s(s+4)(s-3)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s-3} = \frac{A(s+4)(s-3) + Bs(s-3) + Cs(s+4)}{s(s+4)(s-3)}.$$

From the last screen, we have

$$\frac{1}{s(s+4)(s-3)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{(s-3)} = \frac{A(s+4)(s-3) + Bs(s-3) + Cs(s+4)}{s(s+4)(s-3)} \,.$$

Now, equate the two numerators:

$$1 = A(s+4)(s-3) + Bs(s-3) + Cs(s+4).$$

We can find values for A, B and C by choosing "convenient" values for s:

If $s = 0$, then the terms containing B and C vanish. We have

$$1 = A\big((0)+4\big)\big((0)-3\big) \to 1 = -12A, \quad \text{so that} \quad A = -\frac{1}{12} \,.$$

(continued…)

From the last screen, we have $1 = A(s+4)(s-3) + Bs(s-3) + Cs(s+4)$.

If $s = -4$, then the A and C terms vanish, and we have

$$1 = B(-4)\big((-4)-3\big) \to 1 = 28B, \quad \text{so that} \quad B = \frac{1}{28}.$$

If $s = 3$, then the A and B terms vanish, and we have

$$1 = C(3)\big((3)+4\big) \to 1 = 21C, \quad \text{so that} \quad C = \frac{1}{21}.$$

Finally, we have that $L^{-1}\left\{\frac{1}{s(s^2+s-12)}\right\} = L^{-1}\left\{\frac{-1/12}{s} + \frac{1/28}{s+4} + \frac{1/21}{s-3}\right\}$. Thus,

$$y = -\frac{1}{12} + \frac{1}{28}e^{-4t} + \frac{1}{21}e^{3t}.$$