Laplace Transforms

Given a function $y = f(t)$, its Laplace Transform $L\{y(t)\} = H(s)$ is a function in variable s, given by

$$H(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt.$$

For the moment, you don't need to actually use this formula. You just need to memorize a few common results:

Function	**Laplace Transform**
$y = c$ (any constant)	$H(s) = \frac{c}{s}$
$y = t^n$	$H(s) = \frac{n!}{s^{n+1}}$
$y = e^{at}$	$H(s) = \frac{1}{s-a}$
$y = \cos bt$	$H(s) = \frac{s}{s^2+b^2}$
$y = \sin bt$	$H(s) = \frac{b}{s^2+b^2}$
$y = e^{at}\cos bt$	$H(s) = \frac{s-a}{(s-a)^2+b^2}$
$y = e^{at}\sin bt$	$H(s) = \frac{b}{(s-a)^2+b^2}$

If the differential equation has initial conditions $y(0)$ and $y'(0)$, then we have the following two transforms

Function	**Laplace Transform**
y'	$sH(s) - y(0)$
y''	$s^2H(s) - sy(0) - y'(0)$

The Laplace Operator is linear:

$$L\{c_1y_1 + c_2y_2\} = c_1L\{y_1\} + c_2L\{y_2\}.$$

Examples:

$$\text{if } y = 3t^5, \qquad \text{then } H(s) = 3\left(\frac{5!}{s^{5+1}}\right) = \frac{360}{s^6}.$$

$$\text{if } y = e^{4t}, \qquad \text{then } H(s) = \frac{1}{s-4}.$$

$$\text{if } y = 2, \qquad \text{then } H(s) = \frac{2}{s}.$$

$$\text{if } y = \cos 6t, \qquad \text{then } H(s) = \frac{s}{s^2+36}.$$

$$\text{if } y = 4\sin 3t, \qquad \text{then } H(s) = 4\left(\frac{3}{s^2+9}\right) = \frac{12}{s^2+9}.$$

$$\text{if } y = 3e^{-2t}\cos 5t, \qquad \text{then } H(s) = \frac{3(s+2)}{(s+2)^2+25}.$$

A major aspect of Laplace Transforms is to *invert* the transform – to go from the $H(s)$ form back to the function in terms of t:

Example: Find $L^{-1}\left\{\frac{2}{s+5}\right\}$.

Solution: This looks like the form for e^{at} (third one on the last from the previous page), so

$$L^{-1}\left\{\frac{2}{s+5}\right\} = 2e^{-5t}.$$

More Examples of Inversion:

$$L^{-1}\left\{\frac{4}{s}\right\} = 4 \qquad L^{-1}\left\{\frac{6}{s^4}\right\} = t^3 \qquad L^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos 4t \qquad L^{-1}\left\{\frac{6}{(s-1)^2+36}\right\} = e^t \sin 6t$$

You may need to play with constants before you invert:

$$L^{-1}\left\{\frac{5}{s^3}\right\} = \underbrace{5L^{-1}\left\{\frac{1}{s^{2+1}}\right\}}_{\substack{\text{Pull out the 5,}\\ \text{then identify } n.}} = \underbrace{\frac{5}{2}L^{-1}\left\{\frac{2}{s^{2+1}}\right\}}_{\substack{\text{We will need a 2!=2}\\ \text{on top,so balance}\\ \text{outside too.}}} = \frac{5}{2}t^2.$$

$$L^{-1}\left\{\frac{2}{(s-3)^2+7}\right\} = \underbrace{2L^{-1}\left\{\frac{1}{(s-3)^2+\sqrt{7}^2}\right\}}_{\substack{\text{Pull out the 2, and note}\\ \text{that } b=\sqrt{7}.}} = \underbrace{\frac{2}{\sqrt{7}}L^{-1}\left\{\frac{\sqrt{7}}{(s-3)^2+\sqrt{7}^2}\right\}}_{\substack{\text{Balance with a } \sqrt{7} \text{ inside}\\ \text{and outside}}} = \frac{2}{\sqrt{7}}e^{3t}\sin(\sqrt{7}t).$$

Partial fraction decomposition may be needed:

Example: Find $L^{-1}\left\{\frac{1}{s^2-s-12}\right\}$.

Solution: We must decompose $\frac{1}{s^2-s-12}$ first:

$$\frac{1}{s^2-s-12} = \frac{1}{(s-4)(s+3)} = \frac{A}{s-4} + \frac{B}{s+3} = \frac{A(s+3)+B(s-4)}{(s-4)(s+3)}.$$

Thus, comparing numerators, we have

$$1 = A(s+3) + B(s-4).$$

When $s=-3$, we get $1 = B(-7)$, so that $B = -\frac{1}{7}$.
When $s=4$, we get $1 = A(7)$, so that $A = \frac{1}{7}$.

Thus,

$$L^{-1}\left\{\frac{1}{s^2-s-12}\right\} = L^{-1}\left\{\frac{1/7}{s-4} + \frac{-1/7}{s+3}\right\} = \frac{1}{7}L^{-1}\left\{\frac{1}{s-4}\right\} - \frac{1}{7}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{7}e^{4t} - \frac{1}{7}e^{-3t}.$$

So now we have enough basic skills to solve a simple differential equation using Laplace Transforms.

Example: Solve $y'' + 2y' - 15y = 2t,\ y(0) = 1, y'(0) = 0$.

Solution: Take the Laplace Transform of both sides:

$$L\{y'' + 2y' - 15y\} = L\{2t\}$$

Distribute, pull out constants:

$$L\{y''\} + 2L\{y'\} - 15L\{y\} = 2L\{t\}$$

Apply some of the rule of transforms, where $H(s) = L\{y\}$:

$$\underbrace{s^2H(s) - sy(0) - y'(0)}_{L\{y''\}} + 2\underbrace{\left(sH(s) - y(0)\right)}_{L\{y'\}} - 15H(s) = \frac{2}{s^2}$$

Simplify, noting that $y(0) = 1, y'(0) = 0$:

$$s^2H(s) - s \cdot 1 - 0 + 2(sH(s) - 1) - 15H(s) = \frac{2}{s^2}$$
$$s^2H(s) - s + 2sH(s) - 2 - 15H(s) = \frac{2}{s^2}$$

Now, solve for $H(s)$:

$$H(s)[s^2 + 2s - 15] = \frac{2}{s^2} + s + 2$$

Get a common denominator on the right side:

$$H(s)[s^2 + 2s - 15] = \frac{s^3 + 2s^2 + 2}{s^2}$$

Now isolate $H(s)$:

$$H(s) = \frac{s^3 + 2s^2 + 2}{s^2(s^2 + 2s - 15)}.$$

The answer is the inversion of what you see on the right:

$$y = L^{-1}\left\{\frac{s^3 + 2s^2 + 2}{s^2(s^2 + 2s - 15)}\right\}.$$

We'll need to break up this expression using partial fractions. First, factor the denominator:

$$\frac{s^3+2s^2+2}{s^2(s^2+2s-15)} = \frac{s^3+2s^2+2}{s^2(s+5)(s-3)}.$$

Note that the denominator has a linear factor s with multiplicity 2. Thus, the proper decomposition form is:

$$\frac{s^3+2s^2+2}{s^2(s+5)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+5} + \frac{D}{s-3}.$$

Now recompose:

$$\frac{s^3+2s^2+2}{s^2(s+5)(s-3)} = \frac{As(s+5)(s-3) + B(s+5)(s-3) + Cs^2(s-3) + Ds^2(s+5)}{s^2(s+5)(s-3)}.$$

Compare just the numerators:

$$s^3+2s^2+2 = As(s+5)(s-3) + B(s+5)(s-3) + Cs^2(s-3) + Ds^2(s+5).$$

When $s=0$, we have $2 = B(5)(-3)$, so that $B = -\frac{2}{15}$.
When $s=-5$, we have $-73 = C(25)(-8)$, so that $C = \frac{73}{200}$.
When $s=3$, we have $47 = D(9)(8)$, so that $D = \frac{47}{72}$.

Thus, we now have

$$s^3+2s^2+2 = As(s+5)(s-3) - \frac{2}{15}(s+5)(s-3) + \frac{73}{200}s^2(s-3) + \frac{47}{72}s^2(s+5).$$

To find A, use any other value for s. Let's let $s=1$:

$$5 = A(6)(-2) - \frac{2}{15}(6)(-2) + \frac{73}{200}(1)^2(-2) + \frac{47}{72}(1)^2(6).$$

Simplify:

$$5 = -12A + \frac{359}{75}, \qquad \text{which implies } A = -\frac{4}{225}.$$

Finally, we have broken apart the expression $\frac{s^3+2s^2+2}{s^2(s^2+2s-15)}$:

$$\frac{s^3+2s^2+2}{s^2(s^2+2s-15)} = \frac{-4/225}{s} + \frac{-2/15}{s^2} + \frac{73/200}{s+5} + \frac{47/72}{s-3}.$$

Thus,

$$\begin{aligned} y &= L^{-1}\left\{\frac{-4/225}{s} + \frac{-2/15}{s^2} + \frac{73/200}{s+5} + \frac{47/72}{s-3}\right\} \\ &= -\frac{4}{225}L^{-1}\left\{\frac{1}{s}\right\} - \frac{2}{15}L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{73}{200}L^{-1}\left\{\frac{1}{s+5}\right\} + \frac{47}{72}L^{-1}\left\{\frac{1}{s-3}\right\}. \end{aligned}$$

Inverting one at a time, you now have the solution:

$$y = -\frac{4}{225} - \frac{2}{15}t + \frac{73}{200}e^{-5t} + \frac{47}{72}e^{3t}.$$