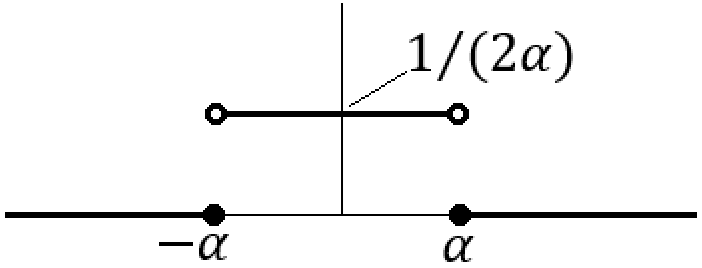
Impulse Forcing Functions

MAT 275

Sometimes, energy is contributed into a system instantaneously. These are called **impulses**.



At right is a function, $y = \begin{cases} 1/(2\alpha), & -\alpha < t < \alpha \\ 0, & t \le -\alpha \text{ or } t \ge \alpha \end{cases}$

The main thing to observe here is that the area under the horizontal bar is 1.

The parameter α can be allowed to trend to 0 as a limit. However, we require that the area below the horizontal bar remain 1 unit. As $|\alpha| \to 0$, the height of the bar trends to infinity. The function is non-zero for increasing smaller amounts of time. In this way, the **unit impulse function** δ is defined as

$$\delta(t) = 0 \quad \text{for all } t \neq 0, \qquad \text{and} \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

The impulse can occur anywhere. If it occurs at $t = c$, we just build in the shift, $\delta(t - c) = 0$ for all $t \neq c$. The integral expression (governing the area) stays the same.

The Laplace Transform of $\delta(t-c)$ is $L\{\delta(t-c)\} = e^{-cs}$.

Example: Find the solution of $y'' + 9y = \delta(t-4)$, with $y(0) = 0$ and $y'(0) = 0$.

Solution: Apply the Laplace Transform operator to both sides:

$$L\{y'' + 9y\} = L\{\delta(t-4)\}$$
$$L\{y''\} + 9L\{y\} = L\{\delta(t-4)\}$$
$$s^2 L\{y\} - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + 9L\{y\} = e^{-4s}$$

$$L\{y\}[s^2 + 9] = e^{-4s}$$

Thus, $L\{y\} = \frac{e^{-4s}}{s^2+9}$, so that the solution is $y = L^{-1}\left\{\frac{e^{-4s}}{s^2+9}\right\}$.

From the last slide, we have that the solution is $y = L^{-1}\left\{\frac{e^{-4s}}{s^2+9}\right\}$.

The e^{-4s} inverts back to $u_4(t)$ … not back to the impulse function!

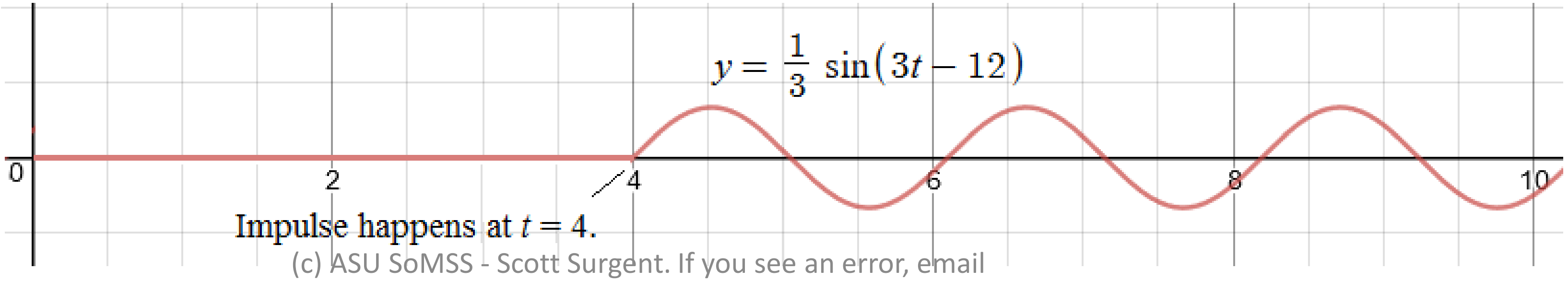
The $\frac{1}{s^2+9}$ inverts as follows: $L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\} = \frac{1}{3}\sin(3t)$.

Put the shift back in.

Thus, the solution is $y = \frac{1}{3}u_4(t)\sin\big(3(t-4)\big)$.

This can be interpreted as follows: When $t < 4$, the factor $u_4(t) = 0$, so $y = 0$, so “nothing is happening”. Then at $t = 4$, an impulse of energy is instantaneously applied to the system, setting in motion the solution, $y = \frac{1}{3}\sin(3t - 12)$, which “starts” at $t = 4$ and continues forever as $t > 4$.

The graph is:



Example: Solve $y'' + 5y' + 6y = \delta(t-3) + \delta(t-6)$, where $y(0) = 0, y'(0) = 0$.

Solution: We have:

$$L\{y''\} + 5L\{y'\} + 6L\{y\} = L\{\delta(t-3)\} + L\{\delta(t-6)\}$$
$$s^2L\{y\} - sy(0) - y'(0) + 5\big(sL\{y\} - y(0)\big) + 6L\{y\} = e^{-3s} + e^{-6s}$$
$$L\{y\}[s^2 + 5s + 6] = e^{-3s} + e^{-6s}$$

Thus,

$$L\{y\} = \frac{e^{-3s} + e^{-6s}}{s^2 + 5s + 6}, \qquad \text{so that} \qquad y = L^{-1}\left\{\frac{e^{-3s} + e^{-6s}}{s^2 + 5s + 6}\right\}.$$

Ignoring the *e* terms for the moment, we have $\frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$.

Next slide…

From the last slide, we have, $y = L^{-1}\left\{\frac{e^{-3s}+e^{-6s}}{s^2+5s+6}\right\}$.

We also have $\frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$.

Inverting, we have $L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$ and $L^{-1}\left\{-\frac{1}{s+3}\right\} = -e^{-3t}$.

Note that the *e* terms invert to $u_3(t)$ and $u_6(t)$. Thus, the solution is

Put the shift back in.

$$y = u_3(t)\left(e^{-2(t-3)} - e^{-3(t-3)}\right) + u_6(t)\left(e^{-2(t-6)} - e^{-3(t-6)}\right).$$

Simplified, we have $y = u_3(t)(e^{6-2t} - e^{9-3t}) + u_6(t)(e^{12-2t} - e^{18-3t})$.

In the graph, there are two "impulses" at t = 3 and t = 6: