Converting from $C_1\cos(bt)+C_2\sin(bt)$ to the $A\cos(bt-P)$ form.

Suppose the solution of a differential equation is

$$y = 0.2\cos(3t) + 0.3\sin(3t).$$

This can be converted into a single cosine function of the form $y = A\cos(3t-P)$, where A is the amplitude and P is the phase shift.

We use the cosine-difference identity, $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$. Let $\alpha = 3t$ and $\beta = P$, then multiply both sides by A:

$$y = A\cos(3t-P) = A(\cos(3t)\cos P + \sin(3t)\sin P).$$

Now equate with the original form of the solution:

$$\begin{aligned} A(\cos(3t)\cos P + \sin(3t)\sin P) &= A\cos(3t)\cos P + A\sin(3t)\sin P \\ &= \underbrace{0.2}_{A\cos P}\cos(3t) + \underbrace{0.3}_{A\sin P}\sin(3t). \end{aligned}$$

Thus, $A\sin P = 0.3$ and $A\cos P = 0.2$, so that $\sin P = \frac{0.3}{A}$ and $\cos P = \frac{0.2}{A}$.

Since $(\cos P)^2 + (\sin P)^2 = 1$, we have

$$\left(\frac{0.2}{A}\right)^2 + \left(\frac{0.3}{A}\right)^2 = 1, \qquad \text{which gives} \qquad 0.2^2 + 0.3^2 = A^2.$$

Thus, $A^2 = 0.13$, or $A = \sqrt{0.13}$. We now have

$$\sin P = \frac{0.3}{A} = \frac{0.3}{\sqrt{0.13}}, \qquad \text{so that} \qquad P = \sin^{-1}\left(\frac{0.3}{\sqrt{0.13}}\right) = 0.98279\ldots \text{radians.}$$

As a check, we have

$$\cos P = \frac{0.2}{A} = \frac{0.2}{\sqrt{0.13}}, \qquad \text{so that} \qquad P = \cos^{-1}\left(\frac{0.2}{\sqrt{0.13}}\right) = 0.98279\ldots \text{radians.}$$

Note that P is also found by $\tan^{-1}\left(\frac{0.3}{0.2}\right) = 0.98279$ radians.

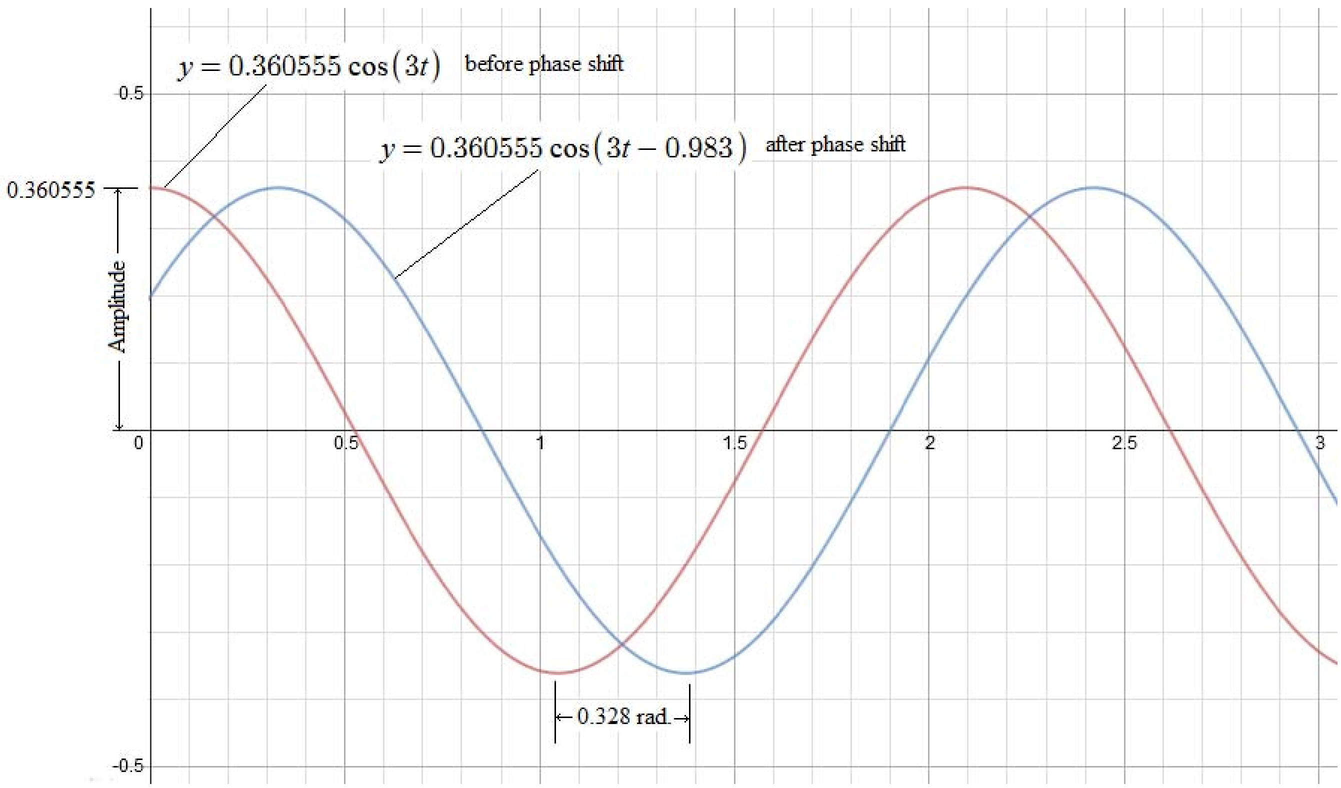
Thus, the original equation, $y = 0.2\cos(3t) + 0.3\sin(3t)$, can now be written as

$$y = \underbrace{\sqrt{0.13}}_{A}\cos(3t - \underbrace{0.98279\ldots}_{P}).$$

The amplitude is $A = \sqrt{0.13} \approx 0.360555$ units, and the phase shift is (rounded) $P = 0.983$ radians. The period is $\frac{2\pi}{b} = \frac{2\pi}{3} \approx 2.09$ units.

You can see the period by comparing the maximum values. Below is the graph of y (in blue). The red curve is the graph of y before the phase shift is included. The red curve has a max at $t = 0$ and again at $t = 2.09$. Shifting the curve horizontally won't affect the period.

Note that $y = 0.360555\cos(3t - 0.983)$ is the same as $y = 0.360555\cos(3(t - 0.328))$. The actual horizontal shift is 0.328 radians.



In general, if a solution is given as

$$y = c_1 \cos bt + c_2 \sin bt,$$

Then the combined cosine form will be

$$y = A\cos(bt - P),$$

Where $A = \sqrt{c_1^2 + c_2^2}$ and $P = \tan^{-1}\left(\frac{c_2}{c_1}\right)$. Just be careful to note in which order the constants are used in the arctangent calculation.