Second-order homogeneous ODEs with constant coefficients: Complex Roots of the Auxiliary Polynomial

MAT 275

Consider the differential equation $y'' + y = 0$.

There are two ways to solve this:

1. By inspection, it appears that $y_1 = \cos x$ and $y_2 = \sin x$ both solve this differential equation. Thus, the general solution is $y = C_1 \cos x + C_2 \sin x$. (We will verify this is true and check for linear independence later on).
2. If we set $y = e^{rx}$, we get $y'' = r^2 e^{rx}$, so that $r^2 e^{rx} + e^{rx} = 0$. After factoring e^{rx}, we have the auxiliary polynomial $r^2 + 1 = 0$. The solutions are complex: $r = \pm i$. Therefore, the solution can be written $y = C_1 e^{ix} + C_2 e^{-ix}$.

The second solution is correct but looks awkward. Furthermore, there should only be one general solution. Are the equations $C_1 e^{ix} + C_2 e^{-ix}$ and $C_1 \cos x + C_2 \sin x$ the same?

Review of Maclaurin Series

Recall that

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots,$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots,$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots.$$

They look similar. Could these be related?

From the last slide, we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots.$$

Now, replace x with ix:

$$e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \frac{1}{6!}(ix)^6 + \frac{1}{7!}(ix)^7 + \frac{1}{8!}(ix)^8 + \cdots.$$

Remember that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$, and so on, repeating the pattern. Thus, the above line is now:

$$e^{ix} = 1 + ix - \frac{1}{2!}x^2 - \frac{1}{3!}ix^3 + \frac{1}{4!}x^4 + \frac{1}{5!}ix^5 - \frac{1}{6!}x^6 - \frac{1}{7!}ix^7 + \frac{1}{8!}x^8 + \cdots.$$

The even-powered terms no longer contain i, while the odd-power terms do.

From the last slide, we had $e^{ix} = 1 + ix - \frac{1}{2!}x^2 - \frac{1}{3!}ix^3 + \frac{1}{4!}x^4 + \frac{1}{5!}ix^5 - \frac{1}{6!}x^6 - \frac{1}{7!}ix^7 + \frac{1}{8!}x^8 + \cdots$.

Now, regroup them and simplify:

$$e^{ix} = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots\right) + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots\right)$$

$$\boxed{e^{ix} = \cos x + i \sin x\,.}$$

Note that $e^{-ix} = \cos x - i\sin x$. Thus, it is reasonable to replace $y = C_1e^{ix} + C_2e^{-ix}$ with $y = C_1(\cos x + i\sin x) + C_2(\cos x - i\sin x)$. This simplifies to

$$y = (C_1 + C_2)\cos x + i(C_1 - C_2)\sin x\,.$$

Since C_1 and C_2 have no meaning yet, we can replace $C_1 + C_2$ with “new” C_1 and $C_1 - C_2$ with “new” C_2. Now we have $y = C_1\cos x + iC_2\sin x$.

From the last slide, we have $y = C_1 \cos x + iC_2 \sin x$ as a solution to $y'' + y = 0$. The imaginary i still remains. Can we get rid of it?

Theorem: If $y = f(x) + ig(x)$ is a solution to an ordinary homogeneous differential equation with constant coefficients, then $f(x)$ and $g(x)$ can be treated as individual solutions.

Proof: Assume that $y = f(x) + ig(x)$ is a solution to $Ay'' + By' + Cy = 0$. Differentiating, we have $y' = f'(x) + ig'(x)$ and $y'' = f''(x) + ig''(x)$. Substituting we have

$$A\big(f''(x) + ig''(x)\big) + B\big(f'(x) + ig'(x)\big) + C\big(f(x) + ig(x)\big) = 0$$
$$\big(Af''(x) + Bf'(x) + Cf(x)\big) + i\big(Ag''(x) + Bg'(x) + Cg(x)\big) = 0.$$

For this to work, we treat 0 as a complex number, $0 + i0$, so that we have

$$\big(Af''(x) + Bf'(x) + Cf(x)\big) + i\big(Ag''(x) + Bg'(x) + Cg(x)\big) = 0 + i0.$$

This implies that $Af''(x) + Bf'(x) + Cf(x) = 0$, so that $f(x)$ is a solution alone, and that $i\big(Ag''(x) + Bg'(x) + Cg(x)\big) = i0$, which is the same as $Ag''(x) + Bg'(x) + Cg(x) = 0$, so that $g(x)$ is also a solution. As long as f and g are linearly independent, they form a general solution.

General rule: If an ODE is of second order, homogeneous and with constant coefficients, and its auxiliary polynomial has complex roots $x = a \pm bi$, then the general solution is

$$y = C_1 e^{ax} \cos(bx) + C_2 e^{ax} \sin(bx)\,.$$

Example: Find the general solution of $y'' + 2y' + 5y = 0$.

Solution: The auxiliary polynomial is $r^2 + 2r + 5 = 0$. Using the quadratic formula, the roots are $y = -1 \pm 2i$, which gives $a = -1$ and $b = 2$. Thus, the general solution is

$$y = C_1 e^{-x} \cos(2x) + C_2 e^{-x} \sin(2x)\,.$$

see... no i

Two things: (1) Assume b is positive so you can ignore the plus-minus, and (2) you no longer need to write the i.

Are they linearly independent? We check the Wronskian (next slide).

From the last slide, we have $y = C_1 e^{-x}\cos(2x) + C_2 e^{-x}\sin(2x)$ and the (possible) general solution of $y'' + 2y' + 5y = 0$. We need to be sure that $y_1 = e^{-x}\cos(2x)$ and $y_2 = e^{-x}\sin(2x)$ are linearly independent. Their derivatives are

$y_1' = e^{-x}(\cos 2x - 2\sin 2x)$ and $y_2' = e^{-x}(2\cos 2x - \sin 2x)$. Thus, we have

$$W(y_1, y_2) = det\begin{bmatrix} e^{-x}\cos(2x) & e^{-x}\sin(2x) \\ e^{-x}(\cos 2x - 2\sin 2x) & e^{-x}(2\cos 2x - \sin 2x) \end{bmatrix}$$

$$= e^{-2x}(2\cos^2 2x - \cos 2x \sin 2x) - e^{-2x}(-2\sin^2 2x - \cos 2x \sin 2x)$$

Trig identity:
$2\cos^2(2x) + 2\sin^2(2x)$
$= 2(\cos^2(2x) + \sin^2(2x))$
$= 2(1) = 2.$

$$= e^{-2x}(2\cos^2 2x + 2\sin^2 2x - \cos 2x \sin 2x + \cos 2x \sin 2x)$$

These terms cancel.

$$= 2e^{-2x}.$$

The important thing is that $2e^{-2x}$ is never 0, so that $e^{-x}\cos(2x)$ and $e^{-x}\sin(2x)$ **are** linearly independent, and that $y = C_1 e^{-x}\cos(2x) + C_2 e^{-x}\sin(2x)$ **is** the general solution of $y'' + 2y' + 5y = 0$.

In general, $e^{ax}\cos bx$ and $e^{ax}\sin bx$ are always linearly independent.

Example: Find the general solution of $y''' - 8y = 0$.

Solution: The auxiliary polynomial is $r^3 - 8 = 0$. The standard factor form of $r^3 - k^3$ is $(r - k)(r^2 + rk + k^2)$. Thus, we can simplify $r^3 - 8 = 0$ as $(r - 2)(r^2 + 2r + 4) = 0$.

The factor $(r - 2)$ gives a root of $r = 2$, so that $y = e^{2x}$ is one solution.

The roots of the factor $(r^2 + 2r + 4)$ are found by the quadratic formula. The roots are $r = -1 \pm i\sqrt{3}$ (Remember, you can ignore the plus-minus). Thus, we have $a = -1$ and $b = \sqrt{3}$, and another solution is $y = e^{-x}\cos(\sqrt{3}x) + e^{-x}\sin(\sqrt{3}x)$.

Combined, the general solution is

$$y = C_1 e^{2x} + C_2 e^{-x}\cos(\sqrt{3}x) + C_3 e^{-x}\sin(\sqrt{3}x).$$

Example: Find the solution to the IVP $y^{iv} - y = 0$, with initial conditions $y(0) = 1, y'(0) = 2, y''(0) = -1,$ and $y'''(0) = 1$.

Solution: The auxiliary polynomial is $r^4 - 1$, which factors to $(r^2 - 1)(r^2 + 1)$ and then again to $(r + 1)(r - 1)(r^2 + 1)$.

- From $(r + 1)$, we get a root $r = -1$, so that $y = e^{-x}$ is a solution.
- From $(r - 1)$, we get a root $r = 1$, so that $y = e^{x}$ is a solution.
- From $(r^2 + 1)$, we get the complex solutions $y = \pm i$, or $0 \pm 1i$, so that $a = 0$ and $b = 1$. Thus, we get $y = e^{0x}\cos(1x) + e^{0x}\sin(1x)$ as solutions, which can be simplified to $y = \cos x + \sin x$.

Thus, the general solution is

$$y = C_1 e^{-x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x\,.$$

From the last slide, we have $y = C_1 e^{-x} + C_2 e^{x} + C_3 \cos x + C_4 \sin x$ as the general solution of the IVP $y^{iv} - y = 0$, with initial conditions $y(0) = 1, y'(0) = 2, y''(0) = -1,$ and $y'''(0) = 1$.

To find the particular solution, we need to differentiate three times:

$$y' = -C_1 e^{-x} + C_2 e^{x} - C_3 \sin x + C_4 \cos x\,,$$
$$y'' = C_1 e^{-x} + C_2 e^{x} - C_3 \cos x - C_4 \sin x\,,$$
$$y''' = -C_1 e^{-x} + C_2 e^{x} + C_3 \sin x - C_4 \cos x\,.$$

The initial conditions are now considered:

$$y(0) = 1 \rightarrow C_1 e^{-0} + C_2 e^{0} + C_3 \cos 0 + C_4 \sin 0 = 1,$$
$$y'(0) = 2 \rightarrow -C_1 e^{-0} + C_2 e^{0} - C_3 \sin 0 + C_4 \cos 0 = 2,$$
$$y''(0) = -1 \rightarrow C_1 e^{-0} + C_2 e^{0} - C_3 \cos 0 - C_4 \sin 0 = -1,$$
$$y'''(0) = 1 \rightarrow -C_1 e^{-0} + C_2 e^{0} + C_3 \sin 0 - C_4 \cos 0 = 1.$$

Recall that $e^0 = 1, \cos 0 = 1$ and $\sin 0 = 0$. Thus, the system from the last slide simplifies to

$$\begin{gathered} C_1 + C_2 + C_3 = 1 \\ -C_1 + C_2 + C_4 = 2 \\ C_1 + C_2 - C_3 = -1 \\ -C_1 + C_2 - C_4 = 1. \end{gathered}$$

Using the matrix-rref feature on the TI-84, the constants are $C_1 = -\frac{3}{4}$, $C_2 = \frac{3}{4}$, $C_3 = 1$ and $C_4 = \frac{1}{2}$. Thus, the solution to the IVP is

$$y = -\frac{3}{4}e^{-x} + \frac{3}{4}e^{x} + \cos x + \frac{1}{2}\sin x\,.$$