Classification of Differential Equations

MAT 275

- Ordinary vs. Partial
  - If the differential equation consists of a function of the form $y = f(x)$ and some combination of its derivatives, then the differential equation is **ordinary**. Note that $y = f(x)$ is a function of a single variable, not a multivariable function.
  - All differential equations in this class are ordinary.
  - In later courses, you may see differential equations with more than one independent variable. These are called partial differential equations.

- Order
  - The **order** of a differential equation is the “highest” derivative of $y = f(x)$ present in the diff. eq.
  - Examples:
    - $y' + 2y = 0$
      - This is first order.
    - $y'' + 2y' - 6y = 1$
      - This is second order.
  - For higher derivatives, we use the notation $y^{(n)}$ to represent the nth derivative of y, rather than write out n prime symbols.

- Linearity
    - A differential equation is **linear** if it can be written in the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

Here, $a_n(x)$ represent functions of x, possibly constants, that are attached to y and its derivatives by multiplication. The term $g(x)$ is not attached to y or its derivatives by multiplication and may be a function of x, or just a constant, possibly 0. The term $g(x)$ is called a **forcing function**.

- Linearity (continued)
  - A linear differential equation <u>does not</u> combine *y* or any of its derivatives through multiplication.
  - Nor does it treat *y* or any of its derivatives as an argument within another operator.
  - Examples:
    - $y' + 2y = 6$ is linear.
    - $y''' + 3y' = -5$ is linear.
    - $(y')^2 + 2y = e^x$ is <u>not</u> linear. Why?
      - Because the y' is being raised to a power other than 1.
    - $y'' \cdot y = 3x$ is <u>not</u> linear. Why?
      - We are multiplying *y* and one of its derivatives.

- Homogeneous (homogeneity)
  - A linear differential equation is **homogeneous** is the forcing function $g(x)$ is 0.
  - Homogeneity only applies to linear differential equations.
  - Examples:
    - $y'' + 2y' = 3y$
      - This is homogeneous because it can be written as $y'' + 2y' - 3y = 0$.
    - $y'' - 4xy + 2x = 0$
      - This is not homogeneous because is can be rewritten as $y'' - 4xy = -2x$. The forcing function is $g(x) = -2x$.

- Autonomous (autonomic)
  - A differential equation is **autonomous (-ic)** if it does not explicitly contain the independent variable.
  - Examples:
    - $y' + 2y = 0$ is autonomous.
    - $y'' + 3y - 4 = 0$ is autonomous.
    - $y''' - 2xy'' + 4y = 0$ is not autonomous.
  - Even though the independent variable may not be explicitly present, it is still "there", implicitly within whatever function is determined to be the solution.
  - Autonomic differential equations are common in population models.

Examples

$$y' + 2xy = 4x$$

It is ordinary. It is first order. It is linear. It is not homogeneous. It is not autonomic.

$$y''(1 - y) = 2x^2$$

It is ordinary. It is second order. It is not linear. Homogeneity is not a concern. It is not autonomic.

Direction Fields

We plot small lines representing slopes at each coordinate (*x*,*y*) in the *xy*-plane. From this, we can infer solution curves.

Example: Sketch a direction field for $y' = x + y$.

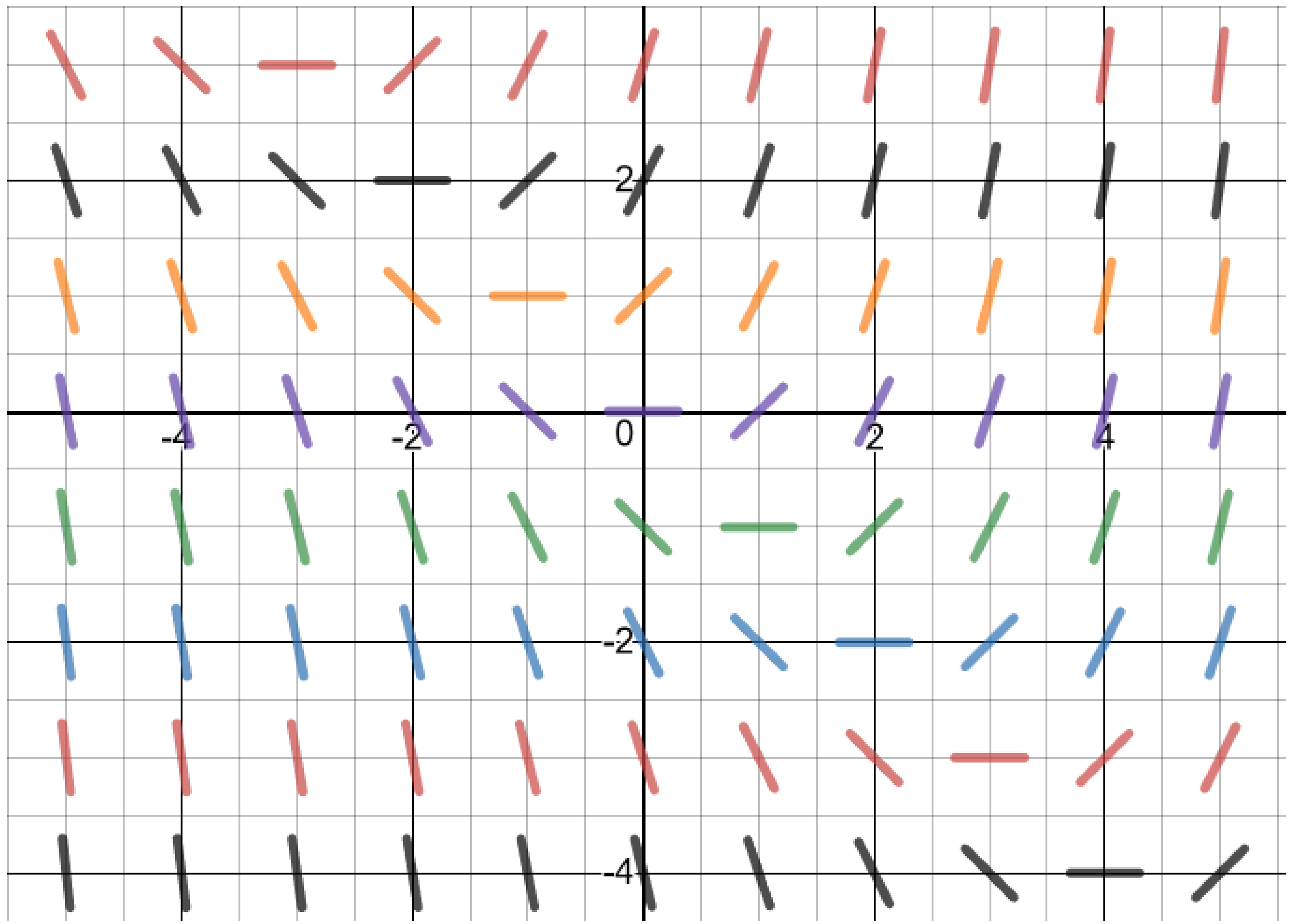
At each (*x*,*y*) coordinate, we determine y'. Some examples are:

At (1,1) we have $y' = 2$.

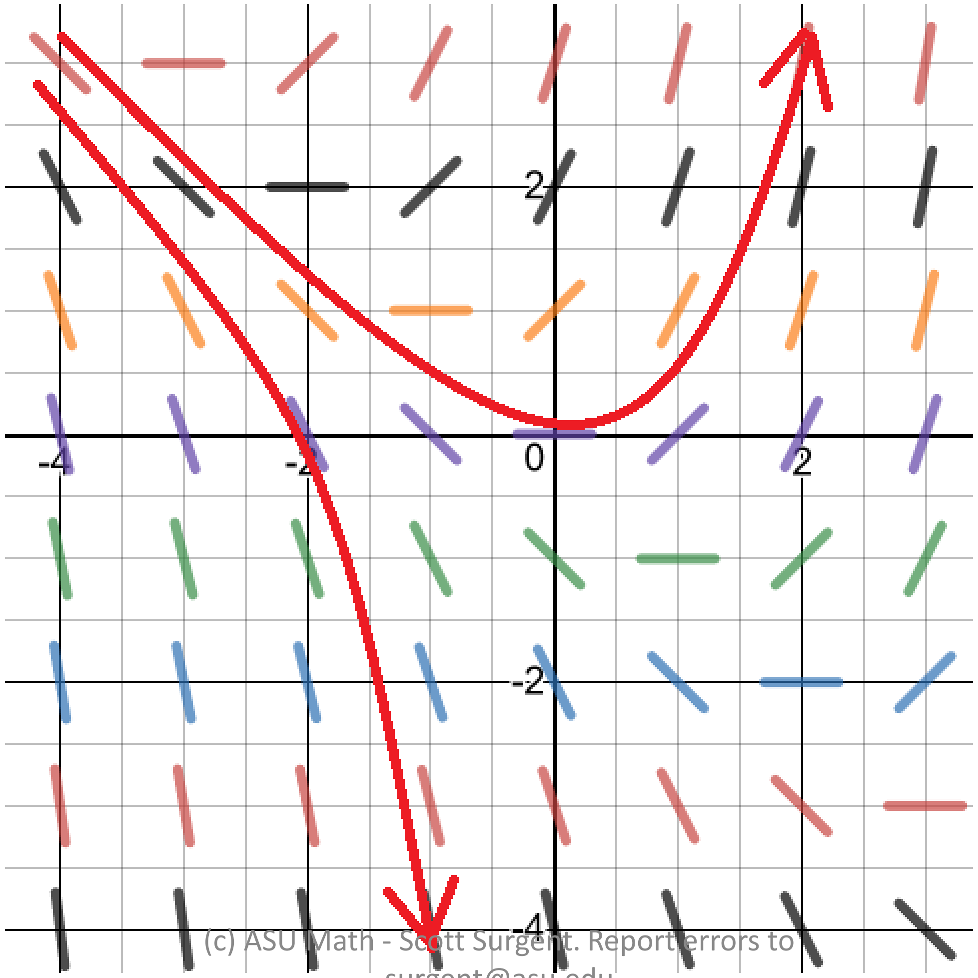
At (2,-3), we have $y' = -1$. And so on.

We do this for "all" possible points in the plane. We get…

Direction field for $y' = x + y$:



We can infer possible solution curves. If you know a point on a particular curve, the rest of the curve can be inferred by "following" the direction field. (We always read left to right, i.e. x is increasing).



Example: Sketch the direction field for $y' = y^3 - 4y$.

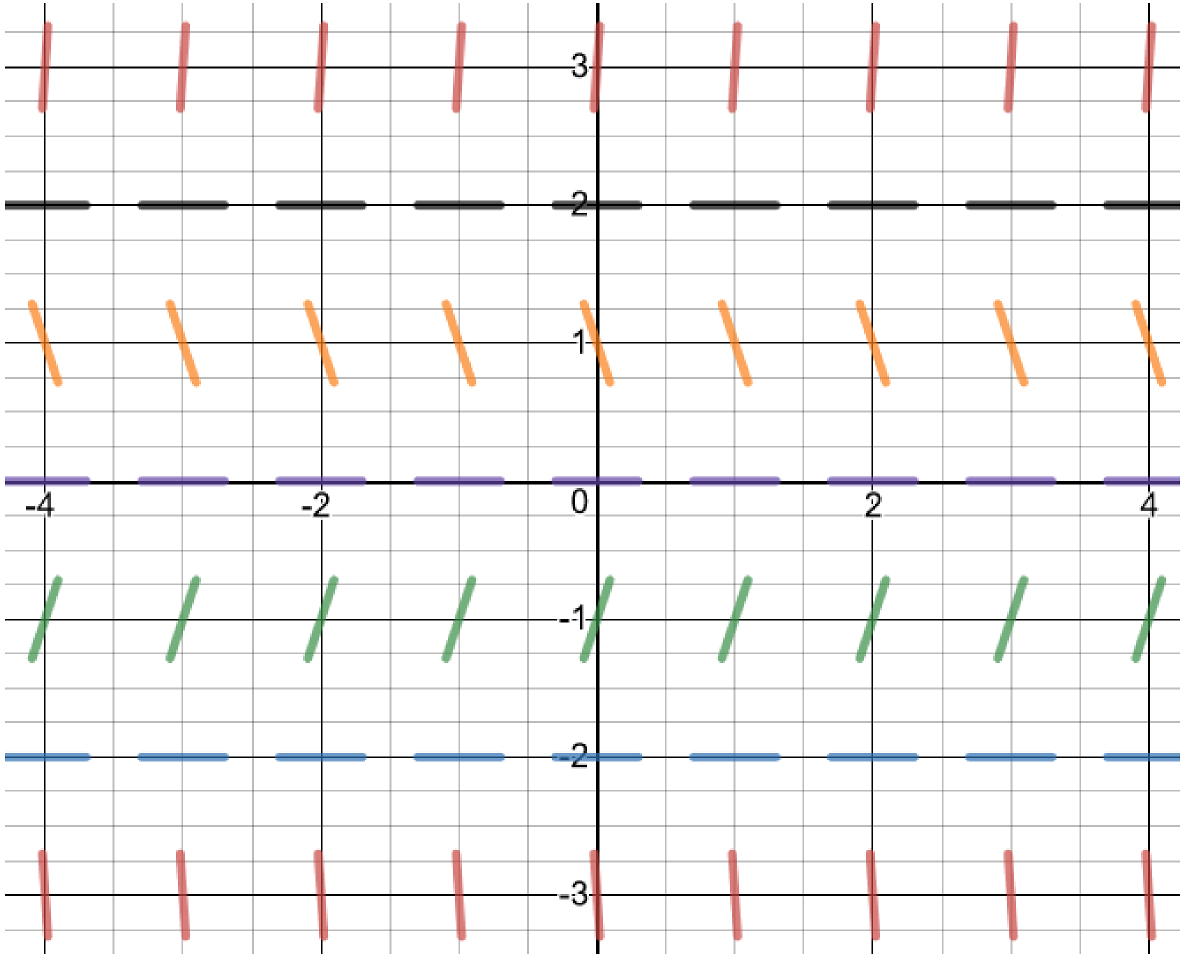
Hint: note that this only depends on *y*. Thus, if we find one slope-line for a particular *y*-value, we can extend it left and right.

Another hint: Note that $y' = 0$ represents a horizontal slope-line and this occurs when $y^3 - 4y = 0$. Factoring, we have

$$y(y^2 - 4) = y(y + 2)(y - 2) = 0.$$

Thus, when $y = 2, -2$ or 0, then $y' = 0$.

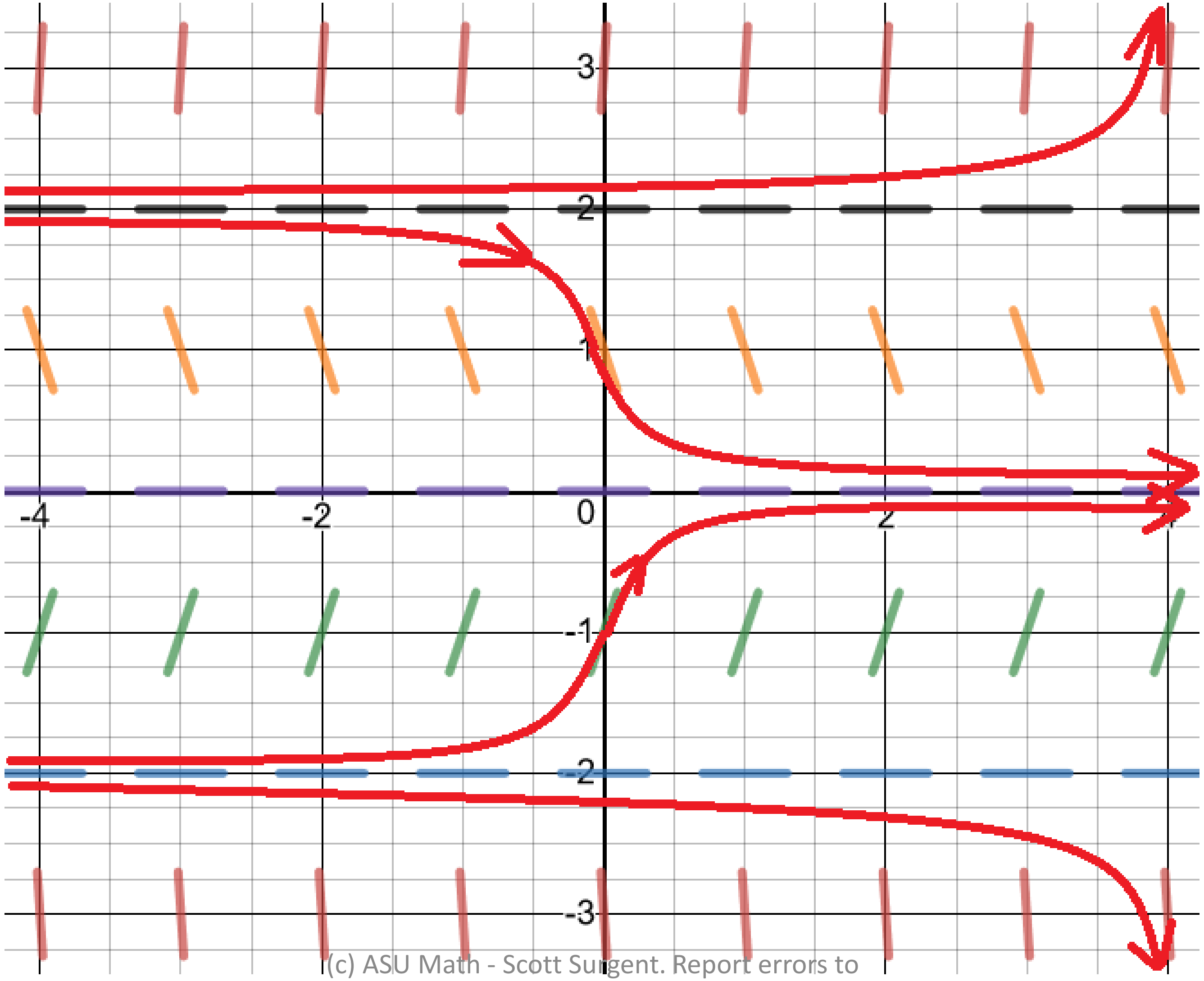
The direction field for $y' = y^3 - 4y$ is:



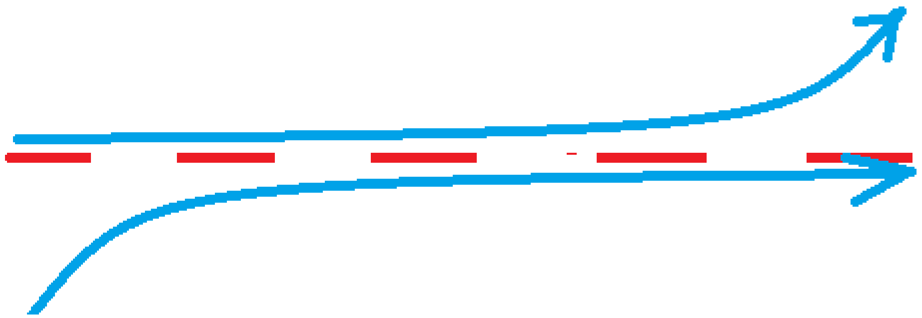
Note the horizontal slopes when y = -2, y = 0 or y = 2. These are **equilibrium** solutions.

Note how the flow lines (representing possible solution curves) veer towards the equilibrium solution $y = 0$. Thus, $y = 0$ is a **stable** equilibrium.

The flow lines veer away from the equilibriums $y = 2$ and $y = -2$. These are **unstable** equilibriums.



A third type of equilibrium is **semi-stable**, in which the solution curves approach the equilibrium asymptotically from one side, yet veer away from the other.



Red: equilibrium solution.
Blue: flow lines.