Initial Value Problems with Discontinuous Forcing Functions

This is meant to expand on Example 1, Section 6.4, in the textbook. They skip a few steps at strategic points, so I wanted to fill in the holes, so to speak.

The differential equation is

$$2y'' + y' + 2y = \begin{cases} 1, & 5 \le t < 20 \\ 0, & t < 5 \text{ or } t \ge 20 \end{cases}, \text{ with } y(0) = 0, y'(0) = 0.$$

The problem is rewritten using the u_c notation:

$$2y'' + y' + 2y = u_5(t) - u_{20}(t).$$

The Laplace Transform Operator is applied to both sides:

$$L\{2y'' + y' + 2y\} = L\{u_5(t) - u_{20}(t)\}$$
$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{u_5(t)\} - L\{u_{20}(t)\}$$
$$2\left(s^2L\{y\} - sy(0) - y'(0)\right) + sL\{y\} - y(0) + 2L\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$
$$2s^2L\{y\} + sL\{y\} + 2L\{y\} = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s} \quad y(0) = 0, y'(0) = 0$$
$$L\{y\}[2s^2 + s + 2] = \frac{e^{-5s}}{s} - \frac{e^{-20s}}{s}$$
$$L\{y\} = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}.$$

What I suggest now is to momentarily "forget" the $e^{-5s} - e^{-20s}$ expression and concentrate on the expression $\frac{1}{s(2s^2+s+2)}$, which needs to be split using partial fractions. Note that $2s^2 + s + 2$ is an irreducible quadratic. We decompose, then recompose:

$$\frac{1}{s(2s^2 + s + 2)} = \frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2} = \frac{A(2s^2 + s + 2) + (Bs + C)s}{s(2s^2 + s + 2)}.$$

Now, equate the numerators:

$$1 = A(2s^2 + s + 2) + (Bs + C)s$$
$$1 = 2As^2 + As + 2A + Bs^2 + Cs$$
$$1 = s^2[2A + B] + s[A + C] + 2A.$$

Since the left side does not have an s^2 or an s term, we assume their coefficients are 0, so we then have that $2A + B = 0$, $A + C = 0$, and $2A = 1$. This forces $A = \frac{1}{2}$, so that $C = -\frac{1}{2}$ and that $B = -1$.

Thus,

$$\frac{1}{s(2s^2+s+2)} = \frac{1/2}{s} + \frac{-s+(-1/2)}{2s^2+s+2}.$$

This is where the book skips about eight steps.

Anyway, we can see the makings of a solution. Since $2s^2+s+2$ does not factor, we must complete the square, and to do that, we need to remove the 2 that is in front of the s^2 first. Then we take half of ½, which is ¼, then square it, giving $\frac{1}{16}$. This is added in and taken out. Note that the last constant is $1-\frac{1}{16}=\frac{15}{16}$:

$$\frac{-s+\left(-\frac{1}{2}\right)}{2s^2+s+2} = \frac{1}{2}\cdot\frac{-s+\left(-\frac{1}{2}\right)}{s^2+\frac{1}{2}s+1} = \frac{1}{2}\cdot\frac{-s+\left(-\frac{1}{2}\right)}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}.$$

The denominator has the portion $\left(s+\frac{1}{4}\right)^2$, so this suggests we need a $\left(s+\frac{1}{4}\right)$ in the numerator in order to do the inverse Laplace transform correctly. So we add it in and take it out:

$$\frac{1}{2}\cdot\frac{-s+\left(-\frac{1}{2}\right)}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = \frac{1}{2}\cdot\frac{-\left(s+\frac{1}{4}-\frac{1}{4}\right)+\left(-\frac{1}{2}\right)}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = \frac{1}{2}\cdot\frac{-\left(s+\frac{1}{4}\right)+\frac{1}{4}+\left(-\frac{1}{2}\right)}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}.$$

In the numerator, distribute the -1 over the pair $\left(s+\frac{1}{4}\right)$ and the $-\frac{1}{4}$, which gives us $\frac{1}{4}$. Now, note that $\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$. So we have:

$$\frac{1}{2}\cdot\frac{-\left(s+\frac{1}{4}\right)-\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = -\frac{1}{2}\cdot\frac{\left(s+\frac{1}{4}\right)+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}.$$

In that last step, I factored the negative to the front.

Remember, the original transform was

$$L\{y\} = \frac{e^{-5s}-e^{-20s}}{s(2s^2+s+2)}.$$

We have now fully split it out as:

$$L\{y\} = (e^{-5s}-e^{-20s})\left[\frac{1/2}{s} - \frac{1}{2}\cdot\frac{(s+1/4)}{(s+1/4)^2+15/16} - \frac{1}{2}\cdot\frac{1/4}{(s+1/4)^2+15/16}\right].$$

That last term is almost in the form for the $e^{at}\sin bt$. Since $b^2 = \frac{15}{16}$, then $b = \frac{\sqrt{15}}{4}$, so we multiply the top by this value, and outside by its reciprocal:

$$L\{y\} = (e^{-5s} - e^{-20s})\left[\frac{1/2}{s} - \frac{1}{2}\cdot\frac{(s+1/4)}{(s+1/4)^2 + 15/16} - \frac{1}{2}\cdot\frac{4}{\sqrt{15}}\cdot\frac{1/4\cdot\sqrt{15}/4}{(s+1/4)^2 + 15/16}\right].$$

We clean that last term up just a tad:

$$L\{y\} = (e^{-5s} - e^{-20s})\left[\frac{1/2}{s} - \frac{1}{2}\cdot\frac{(s+1/4)}{(s+1/4)^2 + 15/16} - \frac{1}{2}\cdot\frac{1}{\sqrt{15}}\cdot\frac{\sqrt{15}/4}{(s+1/4)^2 + 15/16}\right].$$

Ignoring the $(e^{-5s} - e^{-20s})$ expression for the moment, we can see that if we apply the inverse transforms to the expressions in the square brackets, we have:

$$L^{-1}\left\{\frac{1/2}{s}\right\} = \frac{1}{2}, \qquad L^{-1}\left\{-\frac{1}{2}\cdot\frac{(s+1/4)}{(s+1/4)^2 + 15/16}\right\} = -\frac{1}{2}e^{-(1/4)t}\cos\left(\frac{\sqrt{15}}{4}t\right),$$

and

$$L^{-1}\left\{-\frac{1}{2}\cdot\frac{1}{\sqrt{15}}\cdot\frac{\sqrt{15}/4}{(s+1/4)^2 + 15/16}\right\} = -\frac{1}{2\sqrt{15}}e^{-(1/4)t}\sin\left(\frac{\sqrt{15}}{4}t\right).$$

Thus, we have a form of the solution, which we call $h(t)$:

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-(1/4)t}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-(1/4)t}\sin\left(\frac{\sqrt{15}}{4}t\right).$$

Now we bring back in the terms in the expression $(e^{-5s} - e^{-20s})$. Recall that in general,

$$L^{-1}\{e^{-cs}H(s)\} = u_c(t)h(t-c).$$

Here, $H(s)$ is that gigantic expression involving *s* about eight lines above.

Thus,

$$\begin{aligned} L^{-1}\{e^{-5s}H(s)\} &= u_5(t)h(t-5) \\ &= u_5(t)\left[\frac{1}{2} - \frac{1}{2}e^{-(1/4)(t-5)}\cos\left(\frac{\sqrt{15}}{4}(t-5)\right) - \frac{1}{2\sqrt{15}}e^{-(1/4)(t-5)}\sin\left(\frac{\sqrt{15}}{4}(t-5)\right)\right]. \end{aligned}$$

Note that we took the function $h(t)$ and inserted the $(t-5)$ in place of the *t*. That's all we need to do. The $u_5(t)$ simply turns "on" this big shifted function when $t = 5$.

In a similar manner, we have

$$L^{-1}\{e^{-20s}H(s)\} = u_{20}(t)h(t-20)$$
$$= u_{20}(t)\left[\frac{1}{2} - \frac{1}{2}e^{-(1/4)(t-20)}\cos\left(\frac{\sqrt{15}}{4}(t-20)\right) - \frac{1}{2\sqrt{15}}e^{-(1/4)(t-20)}\sin\left(\frac{\sqrt{15}}{4}(t-20)\right)\right].$$

Remember, this all has a leading negative in front because of the expression $(e^{-5s} - e^{-20s})$.

In reality, the only practical way to write such a solution is

$$y = u_5(t)h(t-5) - u_{20}(t)h(t-20),$$

where

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-(1/4)t}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-(1/4)t}\sin\left(\frac{\sqrt{15}}{4}t\right).$$

Then you do the shifting piece by piece as needed, like I did above.