**Sketching a Phase Diagram when the Eigenvalues/Eigenvectors are Complex**

The system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\mathbf{x}$$

has the solution

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}.$$

If we call $\mathbf{u}(t) = e^t \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix}$ and $\mathbf{v}(t) = e^t \begin{bmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$, then, at $t = 0$, we have

$$\mathbf{u}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}(t) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Because the real part of the eigenvalue(s) $1 \pm 2i$ is positive, the "swirls" emanate outward from the origin, and the origin is considered unstable.

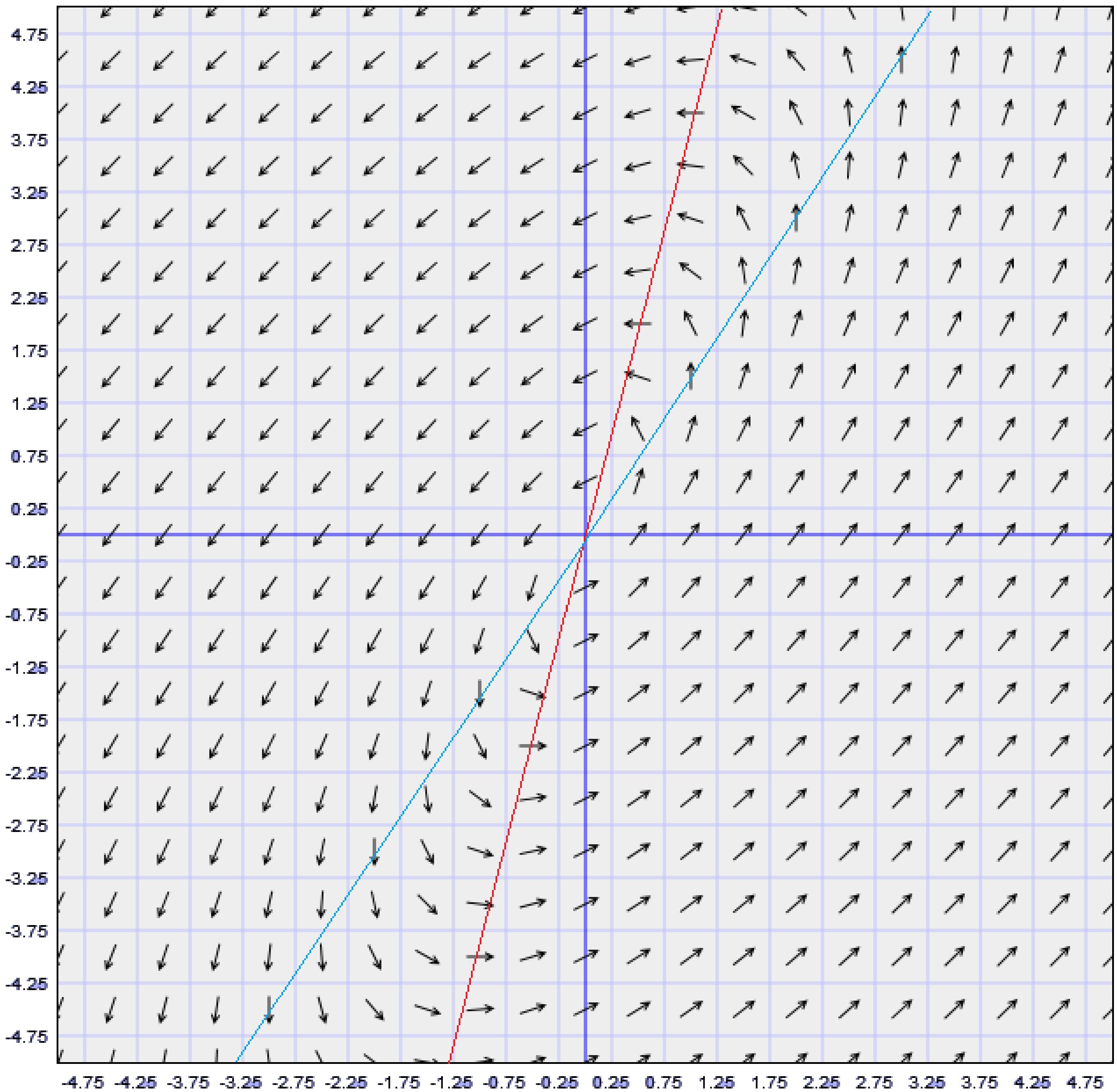
In class, I suggested to test a point (vector) to see if the swirls are clockwise or counterclockwise. I tested $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ by evaluating it in the original differential equation. We get $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, which suggests the contour through $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ continues up and to the right. This would suggest a counterclockwise swirl.

Another way to gain a sense of the pattern is to locate those points (vectors) that have a horizontal component of 0 (they point "up" or "down"), or a vertical component of 0 (they point "left" or "right").

Remember, $\mathbf{x}' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$, so if we use generic $a$ and $b$ to represent points (vectors) in the plane, then the direction lines will be horizontal when the vertical component is 0. That means $x_2'(t) = 0$, and this occurs when $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} h \\ 0 \end{bmatrix} \to \begin{matrix} 3a - 2b = h \\ 4a - b = 0 \end{matrix}$. The bottom equation suggests this happens when $b = 4a$. Similarly, the direction lines will be vertical when the horizontal component is 0, and this happens when $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} \to \begin{matrix} 3a - 2b = 0 \\ 4a - b = k \end{matrix}$, or when $b = \frac{3}{2}a$.

On the next page is a phase diagram generated online at

http://www.math.missouri.edu/~bartonae/pplane.html.




Direction lines are vertical anywhere along the line b = (3/2)a (in blue).

Direction lines are horizontal anywhere along the line b = 4a (in red)

Note that these lines are not asymptotes. They're just guidelines.

Without computer graphics, drawing these by hand can be difficult.