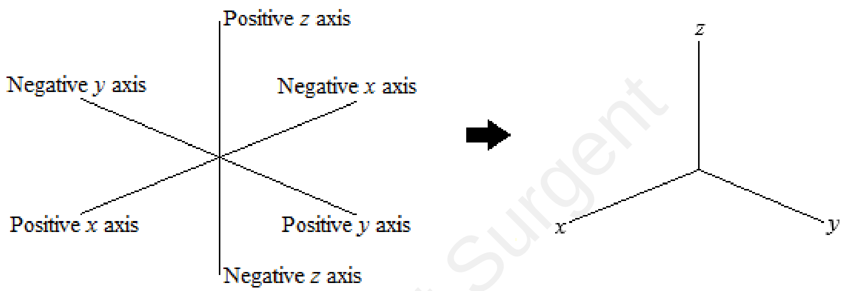
1. The *xyz* Coordinate Axis System

The ***xyz* coordinate axis system**, denoted R^3, is represented by three real number lines meeting at a common point, called the origin. The three number lines are called the ***x*-axis**, the ***y*-axis**, and the ***z*-axis**. Together, the three axes are called the **coordinate axes**.

Perspective (and other forms of artistic license) is used to represent three physical dimensions on a two-dimensional sheet of paper. Below is a common way to represent the three coordinate axes of R^3. At left are the entire three axes with their labels. To the right is a "cleaner" version where only the positive *x*, *y* and *z* axes are drawn. The three axes meet at right angles to one another.



The three axes divide R^3 into eight regions, called **octants**. The region in which *x*, *y* and *z* are positive is called the **first octant** or the **positive octant**. The other octants are not numbered in any conventional way. Negative axes are drawn in only if the problem requires it.

A point is represented by an **ordered triple** (x, y, z), in which from the origin (whose ordered triple is (0,0,0)), one moves *x* units along the *x*-axis, then *y* units parallel to the *y*-axis, and then *z* units parallel to the *z*-axis, to arrive at the point. The values *x*, *y* and *z* are the **coordinates** of the point.

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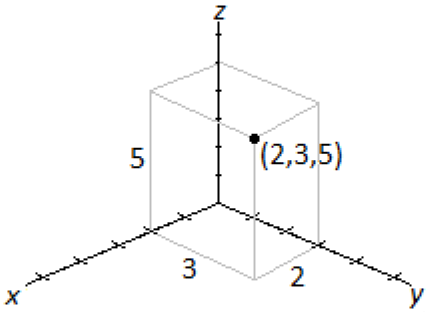
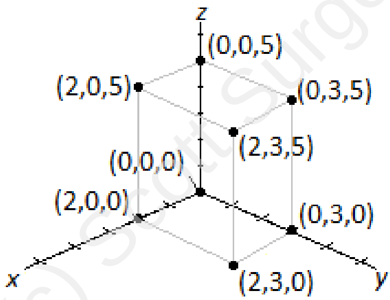
Example 1.1: Represent the point (2,3,5) on an *xyz*-coordinate axis system.

Solution: Due to perspective, we may draw in guidelines to form a "box" in which one corner is the origin, and the opposite corner is the desired point:



Other points are identified to show their relative positions in R^3:



The point (2,3,0) is called a **projection** of (2,3,5) onto the *xy*-plane, found by setting $z = 0$. Other projections can be found similarly.

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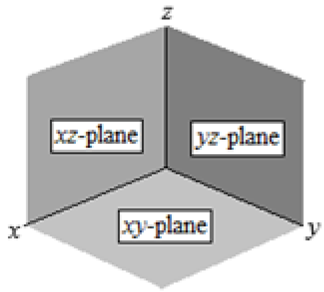
The three coordinate axes, taken two at a time, form three **coordinate planes**.

• The *x*-axis and the *y*-axis form the ***xy*-coordinate plane** and contains points whose ordered triples are of the form $(x, y, 0)$. The equation $z = 0$ represents the *xy*-plane.
• The *x*-axis and the *z*-axis form the ***xz*-coordinate plane** and contains points whose ordered triples are of the form $(x, 0, z)$. The equation $y = 0$ represents the *xz*-plane.
• The *y*-axis and the *z*-axis form the ***yz*-coordinate plane** and contains points whose ordered triples are of the form $(0, y, z)$. The equation $x = 0$ represents the *yz*-plane.



The three coordinate planes.

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Example 1.2: The point (100,6,4) is closest to which coordinate plane?

Solution: Since the *z*-value of 4 is the smallest of the three coordinates, the point (100,6,4) is closest to the *xy* coordinate plane.

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Example 1.3: Given the point $(4,-1,2)$, find its projections onto the *xy*-plane, the *xz*-plane and the *yz*-plane.

Solution: The *xy*-plane is described by the equation $z = 0$, so the projection of $(4,-1,2)$ onto the *xy*-plane is $(4,-1,0)$. Similarly, the projection of $(4,-1,2)$ onto the *xz*-plane is $(4,0,2)$, and $(4,-1,2)$ onto the *yz*-plane is $(0,-1,2)$.

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Example 1.4: Given the point $(4,-1,2)$, find its reflections across the *xy*-plane, the *xz*-plane, the *yz*-plane, and the origin.

Solution: Points reflected across the *xy*-plane are found by negating the *z* coordinate. Thus, the reflection of $(4,-1,2)$ across the *xy*-plane is $(4,-1,-2)$.

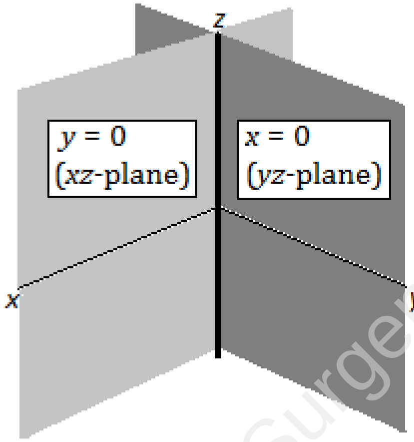
In a similar way, the reflection of $(4,-1,2)$ across the *xz*-plane is $(4,1,2)$, and the reflection of $(4,-1,2)$ across the *yz*-plane is $(-4,-1,2)$.

To reflect across the origin, we negate all three coordinates. This is equivalent to reflecting a point across the *xy*-plane, then the *xz*-plane, then the *yz*-plane (in any order). Thus, the reflection of $(4,-1,2)$ across the origin is $(-4,1,-2)$,

Example 1.5 Describe the intersection of the planes $x = 0$ and $y = 0$.

Solution: The equation $x = 0$ is the *yz*-plane, and the equation $y = 0$ is the *xz*-plane, and they intersect at the *z*-axis. Points on the *z*-axis are described using set notation:

$$\{(x, y, z) \mid x = 0, y = 0, z \in R\}.$$



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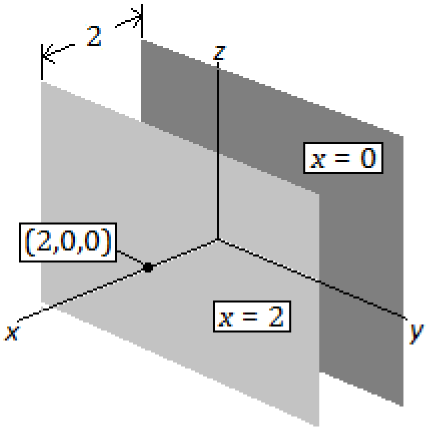
Example 1.6: Describe the equation $x = 2$ as it appears in R^3.

Solution: The equation $x = 2$ includes all points of the form $(2, y, z)$. More generally, it can be described using set notation:

$$\{(x, y, z) | x = 2, y \in R, z \in R\}.$$

It is a plane that is parallel to the *yz*-plane; equivalently, it is the *yz*-plane shifted two units in the positive x direction. Note that the equation $x = 2$ does not imply any restriction on the variables y and z. They can assume any real number value. It is important to remember the "space" in which $x = 2$ is defined. In R^3, it is a plane. In R^2, it would be a vertical line passing through (2,0). In R^1 (or R), it is a point on the real number line.

The graph is on the next page.



The plane x = 2 is parallel to the plane $x = 0$,
shifted 2 units in the positive x direction.

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2. Distance & Midpoint

Given two points $A = (x_0, y_0, z_0)$ and $B = (x_1, y_1, z_1)$ in R^3, the **distance** between A and B is given by

$$D_{A,B} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2},$$

and the **midpoint** between A and B is given by

$$M_{A,B} = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}, \frac{z_0 + z_1}{2}\right).$$

Note that the distance formula is the Pythagorean formula, and that the midpoint formula simply calculates the arithmetic mean (one at a time) of the x-coordinates, the y-coordinates and the z-coordinates.

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Example 2.1: Find the distance from the origin to the point $(3, -1, 5)$.

Solution: The origin is $(0,0,0)$, so the distance is

$$D = \sqrt{(3-0)^2 + (-1-0)^2 + (5-0)^2} = \sqrt{3^2 + (-1)^2 + 5^2} = \sqrt{35}.$$

Example 2.2: Given $A = (-2,1,4)$ and $B = (5,0,-7)$. Find the distance between A and B, and the midpoint of A and B.

Solution: The distance between A and B is

$$\begin{aligned} D_{A,B} &= \sqrt{\big(5-(-2)\big)^2 + (0-1)^2 + (-7-4)^2} \\ &= \sqrt{7^2 + (-1)^2 + (-11)^2} \\ &= \sqrt{171} \\ &\approx 13.077 \text{ units.} \end{aligned}$$

The midpoint between A and B is

$$M_{A,B} = \left(\frac{-2+5}{2}, \frac{1+0}{2}, \frac{4+(-7)}{2}\right) = \left(\frac{3}{2}, \frac{1}{2}, -\frac{3}{2}\right).$$

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Example 2.3: Given $A = (-2,1,4)$ and $B = (5,0,-7)$. Find all points in R^3 that are equidistant from A and B.

Solution: Let $P = (x, y, z)$ represent a point (represented as an ordered triple) equidistant from A and from B. Thus, by the distance formulas, we have

$$D_{P,A} = \sqrt{\big(x-(-2)\big)^2 + (y-1)^2 + (z-4)^2} = \sqrt{(x+2)^2 + (y-1)^2 + (z-4)^2},$$

$$D_{P,B} = \sqrt{(x-5)^2 + (y-0)^2 + \big(z-(-7)\big)^2} = \sqrt{(x-5)^2 + y^2 + (z+7)^2}.$$

Since P is equidistant from A and from B, we have $D_{P,A} = D_{P,B}$. The radicals are squared away, then the binomials expanded by multiplication:

$$\sqrt{(x+2)^2 + (y-1)^2 + (z-4)^2} = \sqrt{(x-5)^2 + y^2 + (z+7)^2}.$$

$$(x+2)^2 + (y-1)^2 + (z-4)^2 = (x-5)^2 + y^2 + (z+7)^2.$$

$$\begin{aligned} &x^2 + 4x + 4 + y^2 - 2y + 1 + z^2 - 8z + 16 \\ &\qquad = x^2 - 10x + 25 + y^2 + z^2 + 14z + 49. \end{aligned}$$

Note that the squared terms cancel one another. We have

$$4x + 4 - 2y + 1 - 8z + 16 = -10x + 25 + 14z + 49.$$

The variable terms are collected to one side and the constant terms to the other:

$$14x - 2y - 22z = 53.$$

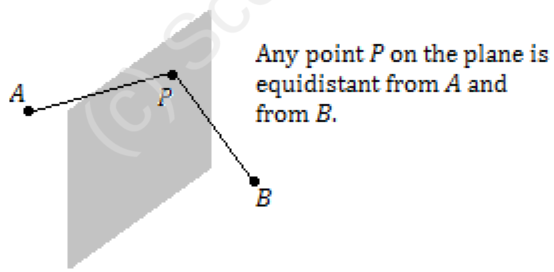
the equation $14x - 2y - 22z = 53$ is true upon substitution by all points that are equidistant from A and B. This forms a plane Q in R^3, which can be written as a set with z is isolated in terms of x and y:

$$Q = \left\{(x, y, z) \mid x \in R, y \in R, z = \frac{7}{11}x - \frac{1}{11}y - \frac{53}{22}\right\}.$$

To check this, we can select arbitrary values for x and y. For example, let $x = 11$ and $y = 22$. This forces $z = \frac{57}{22}$, so a point on Q is $P = \left(11, 22, \frac{57}{22}\right)$. The distance from A to P, and from B to P, are

$$D_{A,P} = \sqrt{(-2-11)^2 + (1-22)^2 + \left(4 - \frac{57}{22}\right)^2} = \sqrt{13^2 + (-21)^2 + \left(\frac{31}{22}\right)^2} \approx 24.738,$$

$$D_{B,P} = \sqrt{(5-11)^2 + (0-22)^2 + \left(-7 - \frac{57}{22}\right)^2} = \sqrt{(-6)^2 + (-22)^2 + \left(-\frac{211}{22}\right)^2} \approx 24.738.$$



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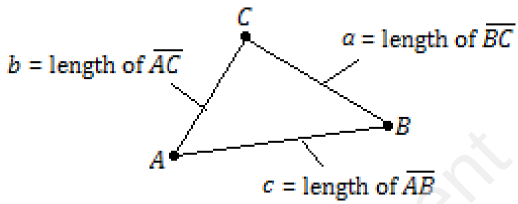
3. Triangles & Collinearity

Three points A, B and C form a **triangle** in that A, B and C are the vertices (corners) of the triangle, and that line segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ form the sides (edges).

Letting a, b and c represent the lengths of the sides of a triangle, and assuming c is the largest of the three values, the **triangle inequality** states that $c \leq a + b$, which simply states that the longest side of a triangle cannot be greater than the sum of the lengths of the two shorter sides:



If $c = a + b$, then the length of the longest side is exactly the sum of the lengths of the two shorter sides, which can only happen when points A, B and C lie on a common line. In such a case, points A, B and C are **collinear**.

The three side-lengths of a triangle are related by the **law of cosines**:

$$c^2 = a^2 + b^2 - 2ab\cos\theta,$$

where c is assumed to be the length of the longest side and θ is the angle formed at point C, where side segments $\overline{AC}$ and $\overline{BC}$ meet. If $\theta = 90°$, then $\cos\theta = 0$, and we have the Pythagorean Formula, which relates the three side-lengths of a **right triangle**:

$$c^2 = a^2 + b^2.$$

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Example 3.1: Show that the points $A = (1,0,2)$, $B = (-2,3,1)$ and $C = (0,4,-2)$ are the vertices of a right triangle.

Solution: Find the lengths of the three sides of the triangle:

$$D_{A,B} = \sqrt{(1-(-2))^2 + (0-3)^2 + (2-1)^2} = \sqrt{3^2 + (-3)^2 + 1^2} = \sqrt{19},$$
$$D_{A,C} = \sqrt{(1-0)^2 + (0-4)^2 + (2-(-2))^2} = \sqrt{1^2 + (-4)^2 + 4^2} = \sqrt{33},$$
$$D_{B,C} = \sqrt{(-2-0)^2 + (3-4)^2 + (1-(-2))^2} = \sqrt{(-2)^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

The length of the segment $\overline{AC}$ is the longest, and we use the Pythagorean Formula:

$$\left(\sqrt{33}\right)^2 = \left(\sqrt{19}\right)^2 + \left(\sqrt{14}\right)^2.$$

Since 33 = 19 + 14 is a true statement, the triangle formed by *A*, *B* and *C* is a right triangle.

When sketching a triangle, we can name the sides as is convenient. In the preceding example, segment $\overline{AC}$ would be given length *b*, if we followed the drawing on the previous page. This is fine, as long as in this case, we remember that *b* is the length of the longest side, and that *c* and *a* are the lengths of the two shorter sides.

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Example 3.2: Show that $A = (2,3,5)$, $B = (6,1,6)$ and $C = (14,-3,8)$ are collinear.

Solution: If *A*, *B* and *C* lie on the same line, then the largest distance between any of the three points will be equal to the sum of the two smaller distances.

The distances are:

$$\begin{aligned} D_{A,B} &= \sqrt{(6-2)^2 + (1-3)^2 + (6-5)^2} \\ &= \sqrt{4^2 + (-2)^2 + 1^2} \\ &= \sqrt{21}, \end{aligned}$$

$$\begin{aligned} D_{A,C} &= \sqrt{(14-2)^2 + (-3-3)^2 + (8-5)^2} \\ &= \sqrt{12^2 + (-6)^2 + 3^2} \\ &= \sqrt{189} \\ &= 3\sqrt{21}, \end{aligned}$$

$$\begin{aligned} D_{B,C} &= \sqrt{(14-6)^2 + (-3-1)^2 + (8-6)^2} \\ &= \sqrt{8^2 + (-4)^2 + 2^2} \\ &= \sqrt{84} = 2\sqrt{21}. \end{aligned}$$

Note that the distance between *A* and *C* is the longest, and that it is the sum of the distance between *A* and *B*, and the distance between *B* and *C*. That is, $3\sqrt{21} = 2\sqrt{21} + \sqrt{21}$, and so we conclude that *A*, *B* and *C* are collinear.

4. Spheres and Ellipsoids

A **sphere** is a set of ordered triples (x, y, z) that are of a fixed distance from a single fixed point (x_0, y_0, z_0), called the **center**, and the distance is called the **radius**, r. Using the distance formula, the simplified formula for a sphere can be written as

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

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Example 4.1: Find the equation of a sphere with center $(2, -1, 9)$ and radius 5.

Solution: The sphere is given by

$$(x - 2)^2 + (y - (-1))^2 + (z - 9)^2 = 5^2,$$

which simplifies to $(x - 2)^2 + (y + 1)^2 + (z - 9)^2 = 25$.

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Example 4.2: Find the equation of a sphere on which the two points $A = (4,1,-1)$ and $B = (6,7,9)$ lie directly opposite one another (that is, the line through them forms a **diameter** of the sphere. Such points are called **antipodal** points).

Solution: The center is the midpoint of *A* and *B*:

$$M_{A,B} = \left(\frac{4+6}{2}, \frac{1+7}{2}, \frac{-1+9}{2}\right) = (5,4,4).$$

The distance from the midpoint to point *A* is:

$$D_{M,A} = \sqrt{(5-4)^2 + (4-1)^2 + \left(4 - (-1)\right)^2} = \sqrt{1^2 + 3^2 + 5^2} = \sqrt{35}.$$

(This is also the distance from the midpoint to *B*.)

This is the radius, and since $r = \sqrt{35}$, then $r^2 = 35$. Thus, the sphere is

$$(x - 5)^2 + (y - 4)^2 + (z - 4)^2 = 35.$$

Example 4.3: Find the equation of the largest possible sphere with center (4,2,5) that is fully contained within the first octant (tangentially "touching" a coordinate plane is permissible).

Solution: The *y*-coordinate of 2 is the smallest of the three coordinates, and is 2 units from the *xz*-coordinate plane. This will be the radius. Thus, the sphere is given by

$$(x-4)^2+(y-2)^2+(z-5)^2=4.$$

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Example 4.4: The sphere $(x+6)^2+(y-1)^2+(z-4)^2=100$ intersects the *yz*-coordinate plane, forming a circle. What is the radius of this circle?

Solution: The *yz*-coordinate plane is given by $x=0$, so we substitute this into the equation of the sphere, and simplify:

$$\begin{aligned}\big((0)+6\big)^2+(y-1)^2+(z-4)^2&=100\\ 6^2+(y-1)^2+(z-4)^2&=100\\ (y-1)^2+(z-4)^2&=64.\end{aligned}$$

The intersection of the sphere with the *yz*-coordinate plane results in a circle of radius $\sqrt{64}=8$.

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A sphere may also be written as $x^2+y^2+z^2+Dx+Ey+Fz=G$, in which case completing the square is needed to rewrite the sphere in simplified form.

Example 4.5: Find the center and radius of the sphere $x^2+2x+y^2-6y+z^2+4z=22$.

Solution: Complete the square three times:

$$\underbrace{x^2+2x+\mathbf{1}}_{(x+1)^2}+\underbrace{y^2-6y+\mathbf{9}}_{(y-3)^2}+\underbrace{z^2+4z+\mathbf{4}}_{(z+2)^2}=\underbrace{22+\mathbf{1}+\mathbf{9}+\mathbf{4}}_{36}.$$

Simplified, we have

$$(x+1)^2+(y-3)^2+(z+2)^2=36.$$

Thus, the sphere has a center of $(-1,3,-2)$ and a radius of $r=\sqrt{36}=6$.

Example 4.6: Explain why $x^2 + y^2 + z^2 + 4x + 6y + 10z + 50 = 0$ cannot represent a sphere.

Solution: Completing the square three times, we have

$$\begin{gathered} x^2 + 4x + 4 + y^2 + 6y + 9 + z^2 + 10x + 25 = -50 + 4 + 9 + 25 \\ (x+2)^2 + (y+3)^2 + (z+5)^2 = -12. \end{gathered}$$

The right side of the equation is negative, while the left side of the equation will always be a non-negative value, so this equation cannot have a solution in R^3. This equation is **inconsistent** (has no solutions).

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An **axis intercept** in R^3 is found by setting two of the variables to 0. Thus, the *x*-axis intercept is given by the ordered triple $(x, 0, 0)$, the *y*-axis intercept is given by the ordered triple $(0, y, 0)$, and the *z*-axis intercept is given by the ordered triple $(0, 0, z)$.

Example 4.7: Find the axis intercepts of the sphere $(x+1)^2 + (y-4)^2 + (z-6)^2 = 41$.

Solution: When $x = 0$ and $y = 0$, we have

$$\begin{aligned} \big((0)+1\big)^2 + \big((0)-4\big)^2 + (z-6)^2 &= 41 \\ 1^2 + (-4)^2 + (z-6)^2 &= 41 \\ 1 + 16 + (z-6)^2 &= 41 \\ (z-6)^2 &= 24 \\ z - 6 &= \pm\sqrt{24} \\ z &= 6 \pm 2\sqrt{6}. \end{aligned}$$

There are two *z*-axis intercepts, at $\big(0,0,6+2\sqrt{6}\big)$ and $\big(0,0,6-2\sqrt{6}\big)$.

When $x = 0$ and $z = 0$, we have

$$\begin{aligned} \big((0)+1\big)^2 + (y-4)^2 + \big((0)-6\big)^2 &= 41 \\ 1^2 + (y-4)^2 + (-6)^2 &= 41 \\ 1 + (y-4)^2 + 36 &= 41 \\ (y-4)^2 &= 4 \\ y - 4 &= \pm 2 \\ y &= 4 \pm 2. \end{aligned}$$

There are two *y*-axis intercepts, at $(0,6,0)$ and $(0,2,0)$.

When $y = 0$ and $z = 0$, we have

$$\begin{aligned}(x+1)^2 + \left((0)-4\right)^2 + \left((0)-6\right)^2 &= 41\\ (x+1)^2 + (-4)^2 + (-6)^2 &= 41\\ (x+1)^2 + 16 + 36 &= 41\\ (x+1)^2 &= -11.\end{aligned}$$

Taking the square root of -11 results in a non-real value. Thus, there are no *x*-axis intercepts.

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Ellipsoids

An **ellipsoid** centered at the origin is written in the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

where $(\pm a, 0,0)$ are the *x*-axis intercepts, $(0, \pm b, 0)$ are the *y*-axis intercepts, and $(0,0, \pm c)$ are the *z*-axis intercepts. The **semi-principal axis radii** are *a*, *b* and *c*, respectively. The semi-principal diameters are 2*a*, 2*b* and 2*c*.

If the ellipsoid is centered at (x_0, y_0, z_0), the equation becomes

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = 1$$

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Example 4.8: Find the axis intercepts of the ellipsoid

$$\frac{x^2}{9} + y^2 + \frac{z^2}{12} = 1.$$

Solution: The *x*-axis intercepts are $(\pm 3,0,0)$, the *y*-axis intercepts are $(0, \pm 1,0)$ and the *z*-axis intercepts are $(0,0, \pm 2\sqrt{3})$. Note that this ellipsoid is centered at the origin.

The semi-principal radii are 3, 1 and $2\sqrt{3}$ units in the direction of the *x*-axis, *y*-axis and *z*-axis, respectively. The semi-principal diameters are twice these figures, or 6, 2 and $4\sqrt{3}$ units in the direction of the *x*-axis, *y*-axis and *z*-axis.

Completing the square may be necessary to determine the ellipsoid's center and axis radii.

Example 4.9: Find the center, the semi-principal axis radii, and the axis intercepts of

$$x^2 + 2y^2 + 4z^2 + 2x - 8y + 24z = -5.$$

Solution: Group the terms by variable, and factor any constants from each grouping:

$$\begin{gathered} x^2 + 2x + 2y^2 - 8y + 4z^2 + 24z = -5 \\ x^2 + 2x + 2(y^2 - 4y) + 4(z^2 + 6z) = -5. \end{gathered}$$

Complete the square three times:

$$x^2 + 2x + \mathbf{1} + 2(y^2 - 4y + \mathbf{4}) + 4(z^2 + 6z + \mathbf{9}) = -5 + \mathbf{1} + \mathbf{8} + \mathbf{36}$$

$$(x+1)^2 + 2(y-2)^2 + 4(z+3)^2 = 40.$$

Note that the 8 on the right side is the "2 times 4" on the left side, and the 36 on the right is the "4 times 9" on the left. Divide now by 40:

$$\frac{(x+1)^2}{40} + \frac{(y-2)^2}{20} + \frac{(z+3)^2}{10} = 1.$$

The ellipsoid's center is $(-1,2,-3)$ and its semi-principal axis radii are $a = \sqrt{40} = 2\sqrt{10}$ in the direction parallel to the *x*-axis, $b = \sqrt{20} = 2\sqrt{5}$ in the direction parallel to the *y*-axis, and $c = \sqrt{10}$ in the direction parallel to the *z*-axis.

For the axis intercepts, we set two variables to 0, and solve for the third variable. For example, to find the *z*-axis intercepts, set *x* = 0 and *y* = 0. This can be done in the original equation:

$$\begin{aligned} x^2 + 2y^2 + 4z^2 + 2x - 8y + 24z &= -5 \\ (0)^2 + 2(0)^2 + 4z^2 + 2(0) - 8(0) + 24z &= -5 \\ 4z^2 + 24z + 5 &= 0. \end{aligned}$$

Using the quadratic formula, we have

$$z = \frac{-24 \pm \sqrt{24^2 - 4(4)(5)}}{2(4)} = \frac{-24 \pm \sqrt{496}}{8} = \frac{-24 \pm 4\sqrt{31}}{8} = -3 \pm \frac{1}{2}\sqrt{31}.$$

Thus, the *z*-axis intercepts are $\left(0,0,-3 \pm \frac{1}{2}\sqrt{31}\right)$. In a similar way, the *y*-axis intercepts are $\left(0,4 \pm \sqrt{6},0\right)$. There are no *x*-axis intercepts (you verify).

5. Multivariable Functions

A function in R^3 has two independent variables, and a third variable dependent on the first two. If x and y represent the independent variables, and z the dependent variable, a **function in two variables** can be written $z = f(x, y)$. Depending on the situation, we can let y be the dependent variable, so that $y = f(x, z)$, or let x be the dependent variable, so that $x = f(y, z)$.

A function in three variables would exist in R^4 and would be written $w = f(x, y, z)$. Its points would be called 4-tuples, written (x, y, z, w). In general, a function in n variables exists in R^{n+1}, has n independent variables and one dependent variable. Any function with n independent variables is called a **multivariable function**, or an $\boldsymbol{n}$**-variable function**. A point in R^n is called an $\boldsymbol{n}$**-tuple**. Note that an n-variable function produces $(n + 1)$-tuples, since the final position will be the dependent variable. We often refer to 2-tuples as *pairs*, 3-tuples as *triples*, and so on.

The **domain** of an n-variable function is the set of ordered n-tuples in R^n for which the function is defined. The **range** is the set of values in R^1 for which the dependent variable can assume. The visual representation of the set of points (ordered n-tuples) for which a function is defined is called a **graph**. In R^3, the graph is often called a **surface**.

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Example 5.1: Given $z = f(x, y) = \frac{1}{x} + 2y$. Find $f\left(\frac{1}{3}, 4\right)$ and the domain of f.

Solution: We have

$$\begin{aligned} f\left(\frac{1}{3}, 4\right) &= \frac{1}{1/3} + 2(4) \\ &= 3 + 8 \\ &= 11. \end{aligned}$$

This is an ordered triple $\left(\frac{1}{3}, 4, 11\right)$ on the graph of f. Note that since x is in the denominator, we must have $x \neq 0$. Thus, the domain is the set of x and y values for which $x \neq 0$. Using set-builder notation, we can write this as

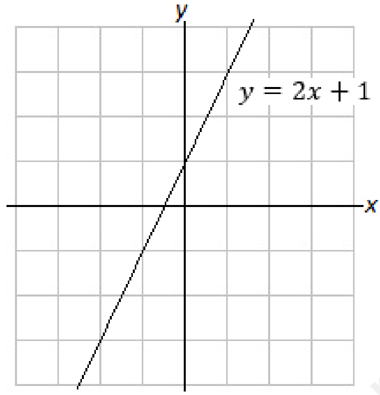
$$\text{Dom } f = \{(x, y) |\ x \in R \text{ and } y \in R \text{ such that } x \neq 0\}.$$

The range can be inferred indirectly. For example, for any z-value, it is possible to find at least one ordered pair (x, y) that produces z. From this, we can state that the range of f is

$$\text{Ran } f = \{z \mid z \in R\}.$$

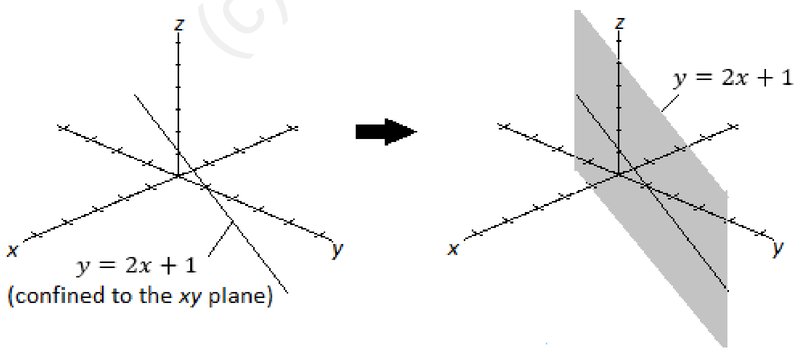
Example 5.2: Describe the graph of $y = 2x + 1$ as it appears in R^3.

Solution: In R^2, this is a line on an *xy*-coordinate axis system with *y*-intercept (0,1) and a slope of 2. It is sketched below:



In R^3, we allow *z* to be any value. Thus, the graph of $y = 2x + 1$ in R^3 is the set of all ordered triples of the form $(x, 2x + 1, z)$. Note that we may choose *x* and *z* independently of one another. However, once *x* is chosen, *y* is then determined by the formula $y = 2x + 1$. Thus, in this example, we would let *x* and *z* be the independent variables, and *y* the dependent variable. The domain is $\{(x, z) |\ x \in R, z \in R\}$ and the range is $\{y | y = 2x + 1\}$.

The graph of $y = 2x + 1$ in R^3 is a plane that extends into the positive and negative *z* directions:



Example 5.3: Find the domain of $z = f(x, y) = \frac{1}{2x-y}$.

Solution: The expression $2x - y$ cannot be zero, $2x - y \neq 0$, or $y \neq 2x$. Using set-builder notation, this is

$$\text{Dom } f = \{(x, y) | \text{ All } x \in R \text{ and } y \in R \text{ such that } y \neq 2x\}.$$

Thus, we may choose x and y independently of one another as long as $y \neq 2x$. For example, $f(3,1)$ is defined, but $f(2,4)$ is not defined. The range is inferred indirectly. If we set $z = 0$, then we have $\frac{1}{2x-y} = 0$. There are no ordered pairs (x, y) that solve this. However, if $z = k$, then $\frac{1}{2x-y} = k$ is solvable. Thus, the range is

$$\text{Ran } f = \{z \mid z \in R \text{ except } z = 0\}.$$

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Determining domain is typically routine, in that we avoid zeros in the denominator, negative values inside an even-index root, and non-positive entries within a logarithm. The table below summarizes domains for common functions.

Type of function	Restrictions on the Domain
n-variable polynomials such as $x^2 + 3x - 1$ or $xy^3 + x^2y - 2x$.	No restrictions.
A radical expression such as $\sqrt[n]{x^3 - 2y}$, where n is an integer ≥ 2. (n is called the *index*)	No restrictions on the expression inside the radical if the index is odd. If the index is even, then the expression must be greater than or equal to 0.
Rational expression such as $\frac{x^2-1}{3x-y^2}$.	The denominator must not equal 0.
Exponential functions such as 2^x or x^y.	The base must be strictly greater than 0, and not equal to 1.
Logarithms such as $\ln(3y - 5x)$.	The expression inside the logarithm must be strictly greater than 0.
Sine and cosine functions.	No restrictions.
Tangent functions.	The expression must not equal $\pm\frac{n\pi}{2}$, where n is an odd integer.

Determining range is not as formulaic. We often use indirect means to infer the domain. For example, we might try setting the function equal to a particular z-value, and work backwards to see it it's possible to solve the equation. If not, then that z-value is outside the range. This method is highly inefficient. Often, the range is inferred by viewing the graph using software.

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Example 5.4: Find the domain and range of $z = g(x, y) = \sqrt{81 - x^2 - y^2}$.

Solution: The expression inside the radical must be non-negative. Thus, we have

$$81 - x^2 - y^2 \geq 0.$$

Rearranging the terms, the domain of g is $\{ (x, y) \mid x^2 + y^2 \leq 81\}$.

The surface of g is a hemisphere of radius 9, and its domain is a filled-in circle of radius 9, centered at (0,0) on the xy-plane. The range of g is $\{z \mid 0 \leq z \leq 9 \}$.

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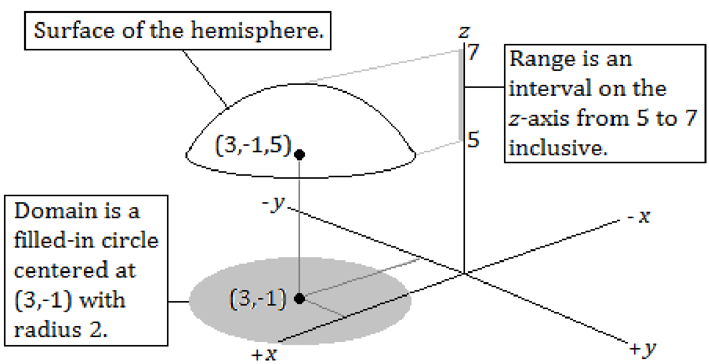
Example 5.5: Find the domain and range of $z = h(x, y) = \sqrt{4 - (x - 3)^2 + (y + 1)^2} + 5$.

Solution: This surface is a hemisphere centered at $(3, -1{,}5)$ with radius 2. It creates a "shadow" onto the xy-plane that is a circle centered at $(3, -1)$ with radius 2. These are the permissible ordered pairs (x, y) that will result in a real-value output z. Thus, the domain of this sphere is

$$\text{Dom } h = \{ (x, y) \mid (x - 3)^2 + (y + 1)^2 \leq 4\}.$$

The range is

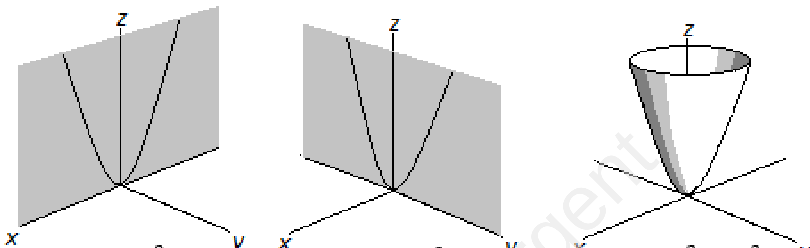
$$\text{Ran } h = \{z \mid 5 \leq z \leq 7 \}.$$



Given a multivariable function $z = f(x, y)$, we can set $x = 0$ and sketch its **trace** on the *yz*-plane, and then set $y = 0$ and sketch its trace on the *xz*-plane. From the two traces, it may be possible to infer the actual surface that results.

Example 5.6: Sketch $z = x^2 + y^2$.

Solution: When $x = 0$, we have $z = y^2$, which is a parabola opening in the positive *z* direction on the *yz*-plane. Similarly, when $y = 0$, we have another parabola $z = x^2$ opening in the positive *z* direction on the *xz*-plane. Together, the two parabola traces suggest that the surface of the function $z = x^2 + y^2$ is a parabolic bowl, or **paraboloid**.



$z = x^2$ (on the *xz*-plane) $z = y^2$ (on the *yz*-plane) $z = x^2 + y^2$

Note that this paraboloid has a vertex at (0,0,0). If positive *z* is considered "up", then we say this paraboloid opens upward. The domain is $\{(x, y)|x \in R, y \in R\}$, and the range is $\{z|z \geq 0\}$.

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Example 5.7: Describe the surface of $z = -x^2 + 4x - y^2 - 2y$.

Solution: Completing the square twice, we have

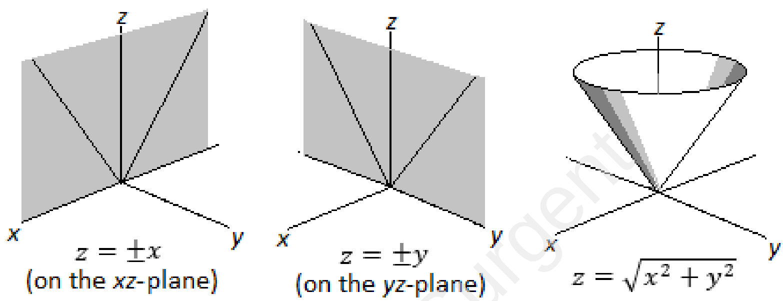
$$\begin{aligned} z &= -x^2 + 4x - y^2 - 2y \\ &= -(x^2 - 4x) - (y^2 + 2y) \\ &= -(x^2 - 4x + 4) - (y^2 + 2y + 1) + 4 + 1 \\ &= -(x - 2)^2 - (y + 1)^2 + 5. \end{aligned}$$

This is a paraboloid that has been shifted 2 units in the *x*-direction, –1 unit in the *y* direction, and 5 units in the *z* direction. The leading negatives in front of the quadratic terms suggest the paraboloid opens in the negative *z* direction. Thus, it has the identical shape as the paraboloid in the previous example, but it has a vertex at $(2, -1, 5)$ and opens "downward". The domain is $\{(x, y)|x \in R, y \in R\}$, and the range is $\{z|z \leq 5\}$.

Example 5.8: Describe the surface $z = \sqrt{x^2 + y^2}$.

Solution: First, note that $x^2 + y^2 \geq 0$ for all x and y, so that the domain is $\{(x, y)|x \in R, y \in R\}$. Note also that the radical results in non-negative values for z, so that the range is $\{z|z \geq 0\}$.

We sketch traces. For example, let $y = 0$, so that means $z = \sqrt{x^2 + 0} = \pm x$. Similarly, when $x = 0$, we have $z = \sqrt{0 + y^2} = \pm y$. These are lines that form a "V" shape in their respective planes. The cross sections parallel to the xy-plane are circles, and together, these facts suggest that $z = \sqrt{x^2 + y^2}$ is a **cone**.



The cone opens in the positive z direction, indicating that the origin is a minimum. The surface given by $z = -\sqrt{x^2 + y^2}$ would be a cone opening in the negative z direction, where the origin would be a maximum, assuming that positive z is "up".

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Example 5.9: A cone with circular cross sections and the vertex at the origin opens in the positive z direction, passing through the point (1,3,7). Find the equation of the cone.

Solution: The general equation of the cone is $z = a\sqrt{x^2 + y^2}$, where a can be determined by evaluating at a known point on the cone's surface. We have

$$7 = a\sqrt{1^2 + 3^2}$$
$$7 = a\sqrt{10}$$
$$a = \frac{7}{\sqrt{10}}.$$

Thus, the cone's equation is $z = \frac{7}{\sqrt{10}}\sqrt{x^2 + y^2} = 7\sqrt{\frac{x^2}{10} + \frac{y^2}{10}}$.

Example 5.10: A cone with circular cross sections and the vertex at the origin opens in the positive z direction, such that the angle at the vertex is $\frac{2\pi}{3}$ radians. Find the equation of the cone.

Solution: Viewing a trace of the cone, we can see the vertex angle. Note that the side of the cone is at an angle of $\frac{\pi}{3}$ radians (half of the vertex angle) from the positive z-axis. From this, we can determine a point on the cone's surface. In the images that follow, we set $y = 0$ and choose $z = 1$. Using a 30-60-90 triangle with shortest leg of length 1, the longer leg is of length $\sqrt{3}$. This is our x value, and the point is $(\sqrt{3}, 0, 1)$. Thus, we have

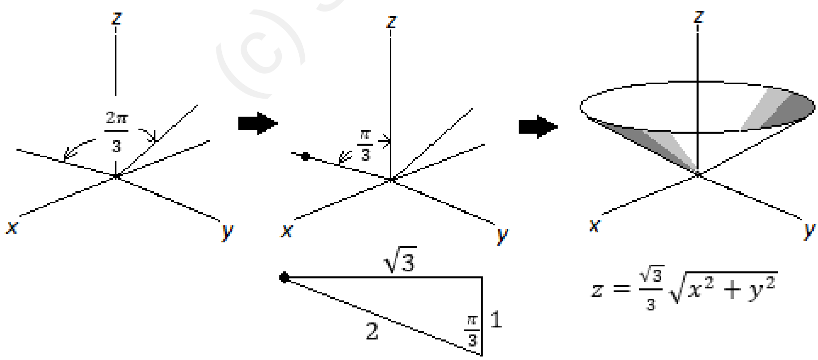
$$z = a\sqrt{x^2 + y^2}$$

$$1 = a\sqrt{\left(\sqrt{3}\right)^2 + 0^2}$$

$$1 = a\sqrt{3}$$

$$a = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3}.$$

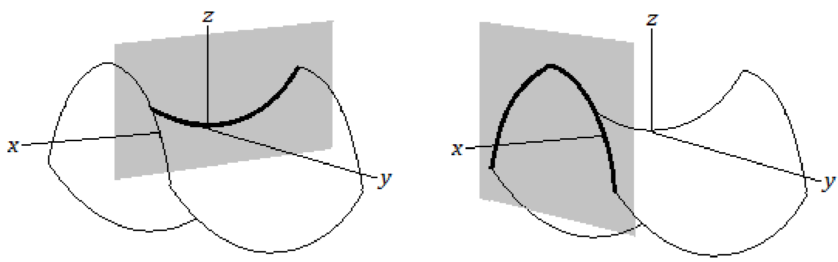
The cone's equation is $z = \frac{\sqrt{3}}{3}\sqrt{x^2 + y^2}$. The images and the final cone are below.



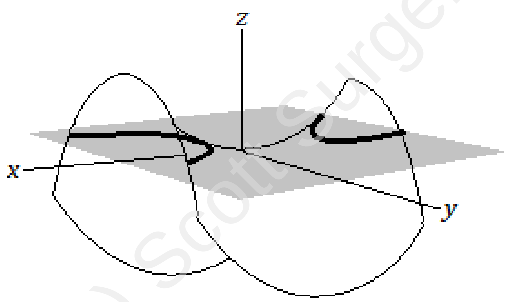
Using right triangles to help find a point on the cone's surface.

Example 5.11: Describe the surface $z = x^2 - y^2$.

Solution: When $y = 0$, the surface's trace on the *xz*-plane is $y = x^2$, a parabola that opens in the positive z direction. When $x = 0$, the trace on the *yz*-plane is $z = -y^2$, a parabola that opens in the negative z direction.

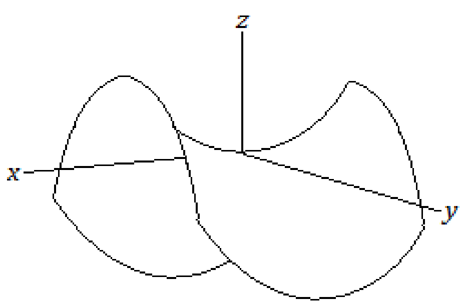


When a plane parallel to the *xz*-plane intersect the surface $z = x^2 - y^2$, it forms a parabola that opens up. When a plane parallel to the *yz*-plane intersects the surface, it forms a parabola that opens down.



When a plane parallel to the *xy*-plane intersects the surface, it forms a hyperbola.

The surface is called a **hyperbolic paraboloid**. It is shaped like a saddle and is informally called a saddle. The origin in this case is the saddle point.



Graph of $z = x^2 - y^2$.

Example 5.12: Describe the surface $x^2 + y^2 - z^2 = 1$.

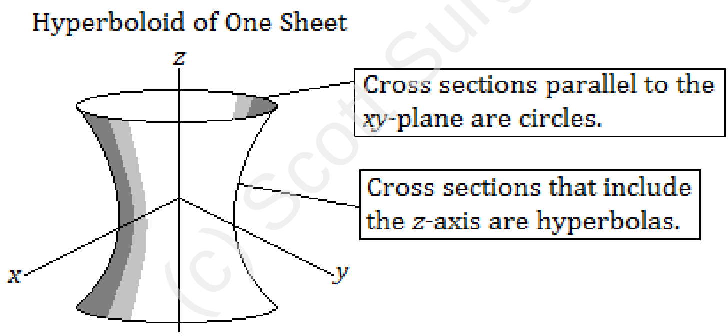
Solution: We can infer the surface's appearance by setting each variable to 0, one at a time.

When $x = 0$, we have $y^2 - z^2 = 1$, which is a hyperbola in the *yz*-plane where the two halves open in the positive and negative *y*-directions.

When $y = 0$, we have $x^2 - z^2 = 1$, which is a hyperbola in the *xz*-plane where the two halves open in the positive and negative *x*-directions.

When $z = 0$, we have $x^2 + y^2 = 1$, which is a circle of radius 1 in the *xy*-plane, centered at the origin.

The resulting shape is called a *hyperboloid of one sheet*. Note that it does not intersect the *z*-axis (the *z*-axis is the axis of symmetry of this surface). Any plane parallel to the *xy*-plane (that is, any plane with the equation $z = k$) will intersect this surface forming a circle. The surface is "narrowest" when $z = 0$.



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Example 5.13: Describe the surface $x^2 - y^2 - z^2 = 1$.

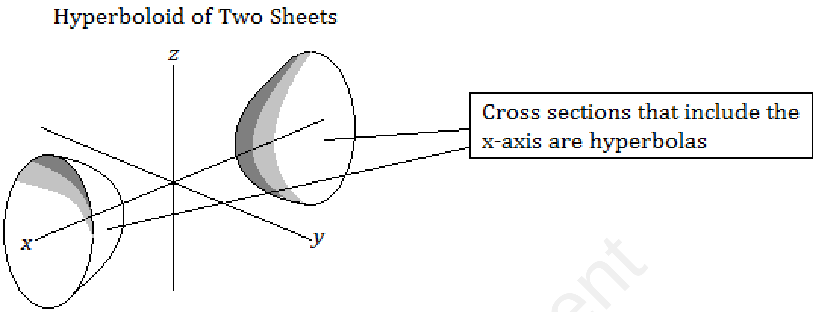
Solution: As in the previous example, we can infer the surface's appearance by setting each variable to 0, one at a time.

When $x = 0$, we have $-y^2 - z^2 = 1$, which has no solution since the left side will always be 0 or negative, while the right side is 1. The surface will not intersect the *yz*-plane.

When $y = 0$, we have $x^2 - z^2 = 1$, which is a hyperbola in the *xz*-plane where the two halves open in the positive and negative *x*-directions.

When $z = 0$, we have $x^2 - y^2 = 1$, which is a hyperbola in the *xy*-plane where the two halves open in the positive and negative *x*-directions.

The resulting shape is called a *hyperboloid of two sheets*. Because it does not intersect the *yz*-plane, the surface is split into two symmetric halves. In fact, it is not difficult to show that the surface is not defined when $-1 < x < 1$.



In this example, the *x*-axis is the axis of symmetry. Any plane that includes the *x*-axis will intersect the surface forming hyperbolas.

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It may be tempting to assume that a hyperboloid of one or two sheets can be "detected" by the number of quadratic terms with a negative coefficient. The next example illustrates how this initial assumption may not be correct.

Example 5.14: Describe the surface $x^2 - 2y^2 - 4z^2 + 2x - 24z + 5 = 0$.

Solution: Completing the square twice (on variables *x* and *z*), we have:

$$x^2 + 2x + 1 - 2y^2 - 4(z^2 - 6z + 9) = -5 + 1 - 36$$
$$(x+1)^2 - 2y^2 - 4(z-3)^2 = -40.$$

Dividing by –40, we have

$$-\frac{(x+1)^2}{40} + \frac{y^2}{20} + \frac{(z-3)^2}{10} = 1.$$

This is a hyperboloid of one sheet, centered at $(-1,0,3)$. Planes parallel to the *yz*-plane will intersect the surface and form ellipses. The axis of symmetry is a line parallel to the *x*-axis, passing through $(-1,0,3)$.

Common Graphs in R^3

Equation	Surface Description
$ax + by + cz = d$	Plane.
$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$	Sphere of radius r and center (x_0, y_0, z_0).
$z = \sqrt{r^2 - x^2 - y^2}$	Hemisphere with a circular base on the *xy*-plane and extending into the positive *z* direction.
$z = x^2 + y^2$	Paraboloid with vertex (0,0,0) opening in the positive *z* direction. The vertex is a minimum.
$z = a\sqrt{x^2 + y^2}$	Cone with vertex at (0,0,0) and opening in the positive *z*-direction, where a is determined by a point on the cone's surface.
$z = -x^2 - y^2$	Paraboloid with vertex (0,0,0) opening in the negative *z* direction. The vertex is a maximum.
$z = (x - x_0)^2 + (y - y_0)^2 + z_0$	Paraboloid with vertex (x_0, y_0, z_0) opening in the positive *z* direction.
$z = x^2 - y^2$ or $z = -x^2 + y^2$	A hyperbolic paraboloid.
$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} - \frac{(z - z_0)^2}{c^2} = 1$	Hyperboloid of one sheet
$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} - \frac{(z - z_0)^2}{c^2} = 1$	Hyperboloid of two sheets.
$y = f(x)$	A "sheet" or "cylinder" in which the curve given by $y = f(x)$ extends into the positive and negative *z* directions, and contains ordered pairs of the form $(x, f(x), z)$.

Surfaces of the general form $Ax^2 + By^2 + Cz^2 + Dx + Ey + Fz + G = 0$ are called **quadric surfaces** in R^3, assuming that A, B and C are not all simultaneously 0 (If they are, then the equation represents a plane). The signs and values of A, B and C determine the type of surface; E, F and G govern shifts in the x, y and z directions simultaneously.

Assuming that the equation is **consistent** (has at least one solution), then some of the common quadric surfaces are spheres, ellipsoids, paraboloids, cones, hyperbolic paraboloids ("saddles"), hyperboloids of one sheet, and hyperboloids of two sheets.

Limits of Functions in R^3

Let $z = f(x, y)$ be a two-variable function in R^3. If x approaches a and y approaches b, then the **general limit** is written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L.$$

For this limit to exist, it must be finite and true for *all* possible paths toward (a, b). If any pair of different paths result in a different limit value, or any one path results in an infinite or undefined limit, then the general limit does not exist.

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Example 5.15: Find the following limit:

$$\lim_{(x,y)\to(1,-2)} (2x^2 y).$$

Solution: For two-variable polynomial terms, the limit will exist and is found by direct evaluation:

$$\lim_{(x,y)\to(1,-2)} (2x^2 y) = 2(1)^2(-2) = -4.$$

Example 5.16: Find the following limit:

$$\lim_{(x,y)\to(3,5)}\left(\frac{x}{y-5}\right).$$

Solution: By direct evaluation, we have

$$\lim_{(x,y)\to(3,5)}\left(\frac{x}{y-5}\right)=\frac{(3)}{(5)-5}=\frac{3}{0}.$$

This is an undefined term. Thus, the limit fails to exist.

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Undefined and Indeterminate: Recall that division by 0 is never allowed. However, depending on the numerator, letting a denominator approach 0 as a limit results in two different situations. If the numerator k is not zero (as a limit), then the expression $\frac{k}{0}$ is *undefined*, such as in Example 5.16. If the numerator is 0 as a limit, then the expression $\frac{0}{0}$ is *indeterminate*, which means that further investigation is needed to determine the limit, if it exists. This is explored in Examples 5.17 and 5.18.

Example 5.17: Find the following limit:

$$\lim_{(x,y)\to(1,1)}\left(\frac{x^2-xy}{x-y}\right).$$

Solution: Evaluation results in the indeterminate form $\frac{0}{0}$:

$$\lim_{(x,y)\to(1,1)}\left(\frac{x^2-xy}{x-y}\right)=\frac{(1)^2-(1)(1)}{(1)-(1)}=\frac{0}{0}.$$

However, we can factor the numerator, then simplify:

$$\frac{x^2-xy}{x-y}=\frac{x(x-y)}{x-y}=x.$$

Re-evaluating the limit, we have

$$\lim_{(x,y)\to(1,1)}\left(\frac{x^2-xy}{x-y}\right)=\lim_{(x,y)\to(1,1)}x=1.$$

Note that the function $z=f(x,y)=\frac{x^2-xy}{x-y}$ is not defined when $y=x$.

Example 5.18: Find the following limit:

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2-y^2}{x^2+y^2}\right).$$

Solution: Evaluation results in the indeterminate form $\frac{0}{0}$:

$$\lim_{(x,y)\to(0,0)}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\frac{(0)^2-(0)^2}{(0)^2+(0)^2}=\frac{0}{0}.$$

The expression is not reducible by factoring. Instead, we try different paths in the *xy*-plane that approach the origin, (0,0). If we can show that two different paths result in two different limits, then the general limit fails to exist.

For the path along the positive *x*-axis towards (0,0), we have $y=0$, so the expression $\frac{x^2-y^2}{x^2+y^2}$ simplifies to

$$\frac{x^2-(0)^2}{x^2+(0)^2}=\frac{x^2}{x^2}=1\ \ (\text{assuming } x\neq 0).$$

Thus, for this particular path, the limit is

$$\lim_{x\to 0}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\lim_{x\to 0}1=1.$$

For the path along the positive *y*-axis towards (0,0), we have $x=0$, so the expression $\frac{x^2-y^2}{x^2+y^2}$ simplifies to

$$\frac{(0)^2-y^2}{(0)^2+y^2}=-\frac{y^2}{y^2}=-1\ \ (\text{assuming } y\neq 0).$$

Thus, the limit for this particular path is

$$\lim_{y\to 0}\left(\frac{x^2-y^2}{x^2+y^2}\right)=\lim_{y\to 0}(-1)=-1.$$

Since two different paths lead to two different limit values, the general limit

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{x^2 + y^2}\right)$$

does not exist. The function $z = f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ is not defined at (0,0), nor does its limit exist as x and y approach (0,0).

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Example 5.19: Find the following limit:

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2 + y^2}\right).$$

Solution: Direct evaluation results in the indeterminate form $\frac{0}{0}$:

$$\lim_{(x,y)\to(0,0)} \left(\frac{xy}{x^2 + y^2}\right) = \frac{(0)(0)}{(0)^2 + (0)^2} = \frac{0}{0}.$$

We try different paths: For a path along the x-axis ($y = 0$), we have

$$\frac{x(0)}{x^2 + (0)^2} = 0, \qquad \text{so that} \quad \lim_{x\to 0} \left(\frac{x(0)}{x^2 + (0)^2}\right) = \lim_{x\to 0}(0) = 0.$$

For a path along the y-axis ($x = 0$), we have

$$\frac{(0)y}{(0)^2 + y^2} = 0, \qquad \text{so that} \quad \lim_{y\to 0} \left(\frac{(0)y}{(0)^2 + y^2}\right) = \lim_{x\to 0}(0) = 0.$$

It might be tempting to infer that since the limit equals 0 along two paths, the general limit would exist and be 0 as well. This is false. Let's try a different path, along the line $y = x$:

$$\lim_{x\to 0} \left(\frac{x(x)}{x^2 + (x)^2}\right) = \frac{x^2}{2x^2} = \frac{1}{2}.$$

We have shown two different paths result in different limit values. Thus, the general limit does not exist.

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