43. Vector Fields

A vector field is a function **F** that assigns to each ordered pair (x, y) in R^2 a vector of the form $\langle M(x, y), N(x, y)\rangle$. We write

$$\mathbf{F}(x, y) = \langle M(x, y), N(x, y)\rangle.$$

This can be extended into higher dimensions. For example. In R^3, we would write

$$\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z)\rangle.$$

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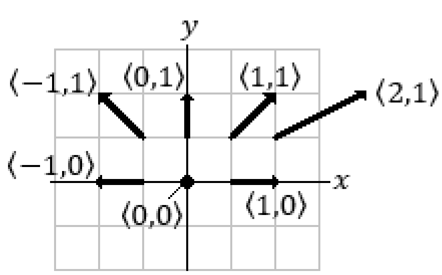
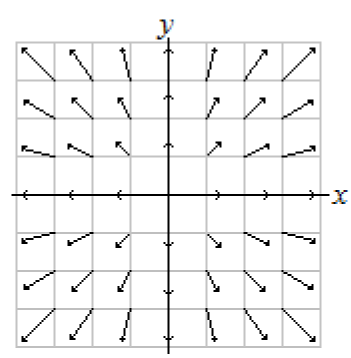
Example 43.1: Sketch $\mathbf{F}(x, y) = \langle x, y\rangle$.

Solution: Using an input-output table, we can show some of the vectors in the vector field **F**:

Ordered pair (x, y)	Vector $\langle x, y\rangle$		Ordered pair (x, y)	Vector $\langle x, y\rangle$
$(0,0)$	$\langle 0,0\rangle$		$(-1,0)$	$\langle -1,0\rangle$
$(1,0)$	$\langle 1,0\rangle$		$(-1,1)$	$\langle -1,1\rangle$
$(1,1)$	$\langle 1,1\rangle$		$(1,-1)$	$\langle 1,-1\rangle$
$(0,1)$	$\langle 0,1\rangle$		$(2,1)$	$\langle 2,1\rangle$
$(1,2)$	$\langle 1,2\rangle$		$(2,2)$	$\langle 2,2\rangle$

The vector $\langle x, y\rangle$ is drawn so that its foot is at the point described by the ordered pair (x, y). Below (left) are a sample of vectors of **F**, and at right, a slightly-more complete rendering of the vector field. In this example, the vectors point radially (along straight lines) away from the origin.





It can be time consuming to sketch a vector field. Also, the vectors themselves “cover up” other vectors, resulting in a cluttered, unreadable image. Certain artistic liberties are allowed. For example, the vectors may be scaled down in size to show relative magnitudes rather than true magnitudes. Often, it is more important to see the “flow” created by the vectors, rather than the actual magnitudes.

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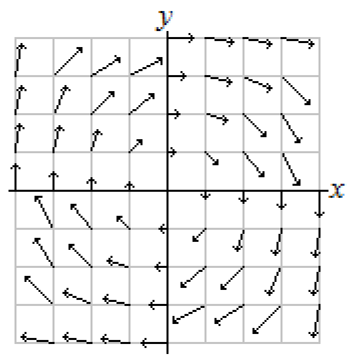
Example 43.2: Sketch $\mathbf{F}(x,y) = \langle y, -x \rangle$.

Solution: An input-output table shows some of the vectors, followed by an image of the vector field.

Ordered pair (x,y)	Vector $\langle x,y \rangle$		Ordered pair (x,y)	Vector $\langle x,y \rangle$
$(0,0)$	$\langle 0,0 \rangle$		$(-1,0)$	$\langle 0,1 \rangle$
$(1,0)$	$\langle 0,-1 \rangle$		$(-1,1)$	$\langle 1,1 \rangle$
$(1,1)$	$\langle 1,-1 \rangle$		$(1,-1)$	$\langle -1,-1 \rangle$
$(0,1)$	$\langle 1,0 \rangle$		$(2,1)$	$\langle 1,-2 \rangle$
$(1,2)$	$\langle 2,-1 \rangle$		$(2,2)$	$\langle 2,-2 \rangle$



The vectors suggest a clockwise rotation around the origin.

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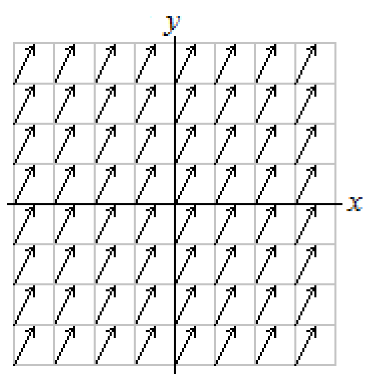
Example 43.3: Sketch $\mathbf{F}(x,y) = \langle 1,2 \rangle$.

Solution: This is a constant vector field. All vectors are identical in magnitude and orientation. In the image below, each vector is shown at half-scale so as not to clutter the image too severely.



This vector field is not radial nor does it suggest any rotation.

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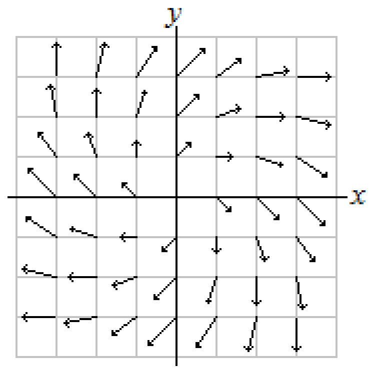
Example 43.4: Sketch $\mathbf{F}(x,y) = \langle x+y, y-x \rangle$.

Solution: The vector field is shown below:



This vector field appears to have both radial and rotational aspects in its appearance.

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Gradient Vector Fields

Given a function $z = f(x,y)$, its gradient is $\nabla f = \langle f_x(x,y), f_y(x,y) \rangle$. This is called a **gradient vector field** (or just **gradient field**). It is also called a **conservative vector field** and is discussed in depth in Section 47. In such a case, the vector field is written as $\mathbf{F}(x,y) = \nabla f = \langle f_x, f_y \rangle$.

Gradient vector fields have an interesting visual property: the vectors in the vector field lie orthogonal to the contours of f.

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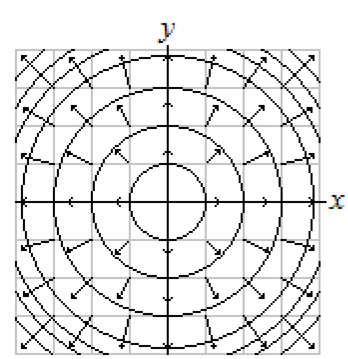
Example 43.5: Given $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$, find $\mathbf{F}(x,y) = \nabla f$ and sketch it along with the contour map of f.

Solution: The vector field is $\mathbf{F}(x,y) = \nabla f = \langle f_x, f_y \rangle = \langle x, y \rangle$. The contours of f are concentric circles of the form $\frac{1}{2}x^2 + \frac{1}{2}y^2 = k$ centered at the origin, the surface being a paraboloid with its vertex at (0,0,0) and opening upward. Note that the vectors in **F** are orthogonal to the contours of f. This is the same vector field as seen in Example 43.1. The vectors point in the direction of increasing z.



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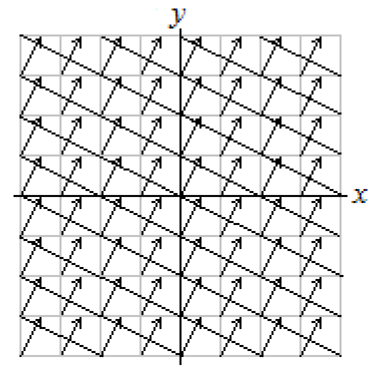
Example 43.6: Given $f(x,y) = x + 2y$, find $\mathbf{F}(x,y) = \nabla f$ and sketch it along with the contour map of f.

Solution: The vector field is $\mathbf{F}(x,y) = \nabla f = \langle f_x, f_y \rangle = \langle 1,2 \rangle$. The surface of f is a plane tilting "upward" as x and y both increase in value. Note that the contours of f are all lines of the form $x + 2y = k$, or $y = -\frac{1}{2}x + \frac{k}{2}$, and that the vectors in **F** are orthogonal to the contours of f, pointing in the direction of increasing z. This is the same vector field as in Example 43.3.



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Not all vector fields are gradient fields. Those in Examples 43.2, and 43.4 are not gradient fields. There do not exist functions $z = f(x,y)$ such that $\mathbf{F}(x,y) = \nabla f$ in these two examples.

If **F** is a gradient field, it is possible to find a function f such that $\mathbf{F}(x,y) = \nabla f$. Such a function f is called a **potential function**, and this is discussed in Section 47.

All constant vector fields $\mathbf{F}(x,y) = \langle a,b \rangle$ are gradient fields, where $f(x,y) = ax + by$ is a potential function. In R^3, we would have $\mathbf{F}(x,y,z) = \langle a,b,c \rangle$, with potential function $f(x,y,z) = ax + by + cz$.

All vector fields of the form $\mathbf{F}(x,y) = \langle M(x), N(y) \rangle$ are gradient fields, where the potential function is $f(x,y) = \int M(x)\,dx + \int N(y)\,dy$.

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Example 43.7: Find the potential functions for $\mathbf{F}(x,y,z) = \langle -1,4,2 \rangle$ and for $\mathbf{G}(x,y) = \langle 2x, y^4 \rangle$.

Solution: For **F**, a potential function is $f(x,y,z) = -x + 4y + 2z$, and for **G**, a potential function is $g(x,y) = \int 2x\,dx + \int y^4\,dy = x^2 + \frac{1}{5}y^5$.

Constants of integration are not necessary. If **F** is a gradient field, then it has infinitely-many potential functions, all equivalent up to its constant of integration. Note that for **F** above, $f(x,y,z) = -x + 4y + 2z + 7$ is also a valid potential function. Usually, we let any such constant be 0.

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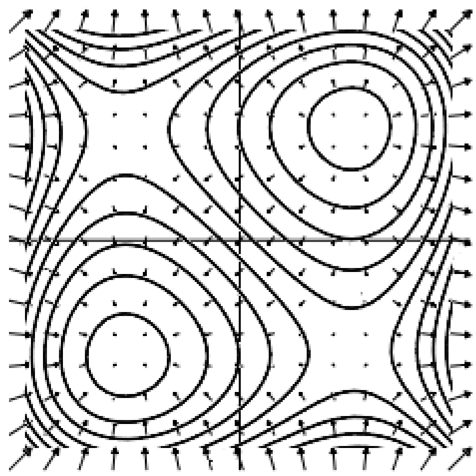
Example 43.8: Given $f(x,y) = x^3 + y^3 - 3x - 3y$, find $\mathbf{F}(x,y) = \nabla f$ and sketch it along with the contour map of f.

Solution: The vector field is

$$\mathbf{F}(x,y) = \nabla f = \langle f_x, f_y \rangle = \langle 3x^2 - 3, 3y^2 - 3 \rangle.$$

The vector field **F** is shown below with the contours of f:



Using techniques of unconstrained optimization (Section 29), there are four critical points. They are: (1,1,–4), a minimum; (–1,–1,4), a maximum; and (1,–1,0) and (–1,1,0), both saddle points. Observe a few things:

- The vectors in **F** always point in the direction of increasing *z*, or "up".
- Note that the vectors point "up" *toward* the maximum at (–1,–1,4) and "up" *away* from the minimum at (1,1,–4).
- At each critical point, $\nabla f = \langle 0,0 \rangle$. In the image above, the vectors near these points have very small magnitudes, while the vectors at these critical points have no magnitude.

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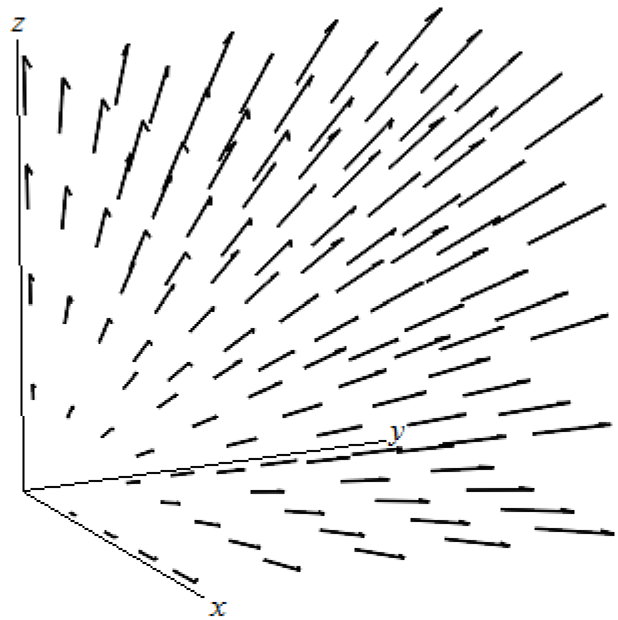
Example 43.9: Sketch $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$.

Solution: The vector field is sketched below, for $x > 0$, $y > 0$ and $z > 0$:



The vectors in **F** all point radially away from the origin, increasing in magnitude the farther away from the origin.

Sketching a vector field in R^3 is nearly impossible to do manually. A computer program is an essential tool to render such fields.

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Given a function $w = f(x, y, z)$, then a gradient field in R^3 can be defined by the gradient of f:

$$\mathbf{F}(x, y, z) = \nabla f = \langle f_x, f_y, f_z \rangle.$$

A function such as $w = f(x, y, z)$ exists in R^4 (three independent variables, one dependent variable). Its contours will be surfaces in R^3, and the vectors in the gradient field given by ∇f will be orthogonal to the contours, which in this case are surfaces.

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Example 43.10: Show that $f(x, y, z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$ is a potential function of $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$. Discuss how the vectors in **F** compare to the contours of f.

Solution: The gradient of f is shown below:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle x, y, z \rangle.$$

The contours of f are found by setting w equal to various constants. For example, when $w = 1$, then we have

$$1 = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2, \qquad \text{so that} \qquad x^2 + y^2 + z^2 = 2.$$

This is a sphere centered at the origin with radius $\sqrt{2}$. In fact, all contours of f are spheres. As w increases in value, its contour at $w = k$ is given by the sphere $x^2 + y^2 + z^2 = 2k$.

For example, the vector whose foot lies at $(1,2,3)$ is given by $\langle 1,2,3 \rangle$. The point $(1,2,3)$ itself lies on a sphere with radius $\sqrt{14}$. The vector $\langle 1,2,3 \rangle$ has its foot on this sphere, oriented orthogonally to this sphere, pointing directly away from the origin (in this case).

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