MAT 267 – Exam 3b – Surgent – April 17, 2025 Name: Key

1. Evaluate
   a. Form A: $\int_0^1 \int_0^2 \int_0^3 (2x+4y)\, dz\, dy\, dx$

Inner: $\int_0^3 (2x+4y)\, dz = [(2x+4y)z]_0^3 = (2x+4y)3 = 6x+12y$
Middle: $\int_0^2 (6x+12y) dy = [(6xy+6y^2)]_0^2 = 12x+24$
Outer: $\int_0^1 (12x+24) dx = [6x^2+24x]_0^1 = 30$

   b. Form B: $\int_0^1 \int_0^2 \int_0^3 (2x+4z)\, dz\, dy\, dx.$

Inner: $\int_0^3 (2x+4z)\, dz = [2xz+2z^2]_0^3 = 2x(3)+2(3)^2-0 = 6x+18$
Middle: $\int_0^2 (6x+18) dy = [(6x+18)y]_0^2 = 2(6x+18) = 12x+36$
Outer: $\int_0^1 (12x+36) dx = [6x^2+36x]_0^1 = 42$

2. Rewrite the triple integral $\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dy\, dx$ in spherical coordinates. Do not solve, just set it up.

Form A: $\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dy\, dx$ is an eighth-sphere of radius 5 above the *xy*-plane in the first octant. Thus, $0 \le \rho \le 5, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}$. Note that $(x^2+y^2+z^2) = \rho^2$. The Jacobian is $\rho^2 \sin\phi$. The integral is $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^5 \rho^2(\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^5 (\rho^4 \sin\phi)\, d\rho\, d\theta\, d\phi$

Form B: $\int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \int_0^{\sqrt{25-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dy\, dx$ is a half-sphere of radius 5 above the *xy*-plane. Thus, $0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}$. Note that $(x^2+y^2+z^2) = \rho^2$. The Jacobian is $\rho^2 \sin\phi$. The integral is $\int_0^{\pi/2} \int_0^{2\pi} \int_0^5 \rho^2(\rho^2 \sin\phi)\, d\rho\, d\theta\, d\phi = \int_0^{\pi/2} \int_0^{2\pi} \int_0^5 (\rho^4 \sin\phi)\, d\rho\, d\theta\, d\phi$

3. Set up a triple integral in the $dz\, dy\, dx$ ordering that gives the volume under the plane with axis intercepts (4,0,0), (0,8,0) and (0,0,10) confined to the first octant. Do not solve, just set it up.

The plane is $\frac{x}{4}+\frac{y}{8}+\frac{z}{10} = 1$ which is $z = 10-\frac{5}{2}x-\frac{5}{4}y$. In the xy-plane, the line connecting (4,0) to (0.8) is $y = 8-2x$. Thus, the integral is $\int_0^4 \int_0^{8-2x} \int_0^{10-\frac{5}{2}x-\frac{5}{4}y} 1\, dz\, dy\, dx$.

4. Evaluate $\int_C x\,ds$
   a. Form A: where C is the portion of $x^2+y^2=9$ from (3,0) to (0,3).

Parameterize: $\mathbf{r}(t)=\langle 3\cos t, 3\sin t\rangle, 0\le t\le \frac{\pi}{2}$, so that $\mathbf{r}'(t)=\langle -3\sin t, 3\cos t\rangle$ and $ds=|\mathbf{r}'(t)|dt=\sqrt{(-3\sin t)^2+(3\cos t)^2}=\sqrt{9(\sin^2 t+\cos^2 t)}=\sqrt{9}=3\,dt$. The integral is $\int_0^{\pi/2}(3\cos t)\,3\,dt=3(3\sin t)_0^{\pi/2}=3(3)=9$.

   b. Form A: where C is the line from (3,0) to (0,3).

Parameterize: $\mathbf{r}(t)=\langle 3-3t, 3t\rangle, 0\le t\le 1$, so that $\mathbf{r}'(t)=\langle -3,3\rangle$ and $ds=|\mathbf{r}'(t)|dt=\sqrt{(-3)^2+3^2}=\sqrt{18}\,dt$. The integral is $\int_0^1(3-3t)\sqrt{18}dt=\sqrt{18}\left(3t-\frac{3}{2}t^2\right)_0^1=\sqrt{18}\left(3-\frac{3}{2}\right)=\frac{3}{2}\sqrt{18}=\frac{9}{2}\sqrt{2}$

5. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y)=\langle 4x^3y+2, x^4+3y^2\rangle$ and C is the path from (0,0) to (4,3) to (7,5) to (2,1).

It is conservative because $M_y=4x^3, N_x=4x^3$ so that $M_y=N_x$. Use the FTLI.
The potential function is $f(x,y)=x^4y+2x+y^3$.
Form A: Only the endpoints matter. $f(1,2)=(1)^4(2)+2(1)+(2)^3=12$, $f(0,0)=0$, and $12-0=12$.

Form B: Only the endpoints matter. $f(2,1)=(2)^4(1)+2(2)+(1)^3=21$, $f(0,0)=0$, and $21-0=21$.

6. Use Green's Theorem to find the work performed by the vector field $\mathbf{F}(x,y)=\langle y^2, 2x+y\rangle$ on a particle moving along the path from (0,0) to (1,0) to (1,4) back to (0,0). For full credit, show the integral with the proper bounds and integrand.

Integrand is $N_x-M_y=2-2y$.
Region is a triangle with x-axis as a leg ($y=0$), the line $y=4x$ as the hypotenuse, and $x=1$ as the other leg.
The integral is $\int_0^1\int_0^{4x}(2-2y)\;dy\,dx$. When evaluated, it is $-4/3$.

7. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y)=\langle 2x+y, y^2+3\rangle$ and C is the path $y=x^2$ from (0,0) to (3,9).

Form A: Not conservative, not a path, so parameterize: $\mathbf{r}(t)=\langle t,t^2\rangle, 0\le t\le 4$, so that $\mathbf{r}'(t)=\langle 1,2t\rangle$.
Rewrite $\mathbf{F}$ in terms of t: $\mathbf{F}(t)=\langle 2(t)+(t^2),(t^2)^2+3\rangle=\langle t^2+2t, t^4+3\rangle$.
Dot: $\mathbf{F}\cdot d\mathbf{r}=\langle t^2+2t,t^4+3\rangle\cdot\langle 1,2t\rangle=t^2+2t+2t^5+6t=2t^5+t^2+8t$.
Therefore, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is $\int_0^4(2t^5+t^2+8t)dt=\left[\frac{1}{3}t^6+\frac{1}{3}t^3+4t^2\right]_0^4=\frac{4352}{3}=1450.666\ldots$.

Not conservative, not a path, so parameterize: $r(t)=\langle t,t^2\rangle, 0\le t\le 3$, so that $r'(t)=\langle 1,2t\rangle$.
Rewrite F in terms of t: $F(t)=\langle 2(t)+(t^2),(t^2)^2+3\rangle=\langle t^2+2t,t^4+3\rangle$.
Dot: $F\cdot dr=\langle t^2+2t,t^4+3\rangle\cdot\langle 1,2t\rangle=t^2+2t+2t^5+6t=2t^5+t^2+8t$.
Therefore, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is $\int_0^3(2t^5+t^2+8t)dt=\left[\frac{1}{3}t^6+\frac{1}{3}t^3+4t^2\right]_0^3=243+9+36=288$.

8. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle 5y + e^x, 12x + \cos(7y) \rangle$ and C is a path from (4,3) to (7,9) to (9,3) back to (4,3).

Both forms: $N_x - M_y = 12 - 5 = 7$, so using Greens, we have $\iint_R 7\, dA = 7 \iint_R dA$. The region is a triangle with base 5, height 6, so the area is $\frac{1}{2}(6)(5) = 15$, so that $7 \iint_R dA = 7(15) = 105$.

In Form A, the path was traversed counterclockwise, so this is the answer. In form B, the path was traversed clockwise, so the answer is negated: -105.

9. Solid S is bounded below by the paraboloid $z = x^2 + y^2$ and above by the paraboloid $z = 8 - x^2 - y^2$. Set up a triple integral in cylindrical coordinates that gives the volume of S. Do not solve, just set it up.

Setting the shapes equal gives $x^2 + y^2 = 8 - x^2 - y^2 \rightarrow 0 = 8 - 2(x^2 - y^2) \rightarrow 4 = x^2 + y^2$. The region of integration is a circle of radius 2. Converting to polar, we have $0 \le t \le 2, 0 \le \theta \le 2\pi$. The integral is $\int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} dz\, r\, dr\, d\theta$.

10. Let $\mathbf{F}(x, y) = \langle 5x^4y^3,\ 3x^5y^2 \rangle$. Suppose path C is a polygon that begins and ends at the origin and passes through (3,5).

a) Evaluate $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is two adjacent sides of the polygon, from (0,0) to (1,0) to (3,5).

Both forms: the vector field is conservative and the potential function is $f(x, y) = x^5y^3$. Using the FTLI from (0,0) to (3,5), we have $f(3,5) = 3^5 5^3 = 30{,}375$ and $f(0,0) = 0$, so that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 30{,}375 - 0 = 30{,}375$. For Form A, the point was (2,5), so the answer was 4000,

b) Using your answer from part (a), find the value of $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_2 is the other three sides of the polygon from (3,5) to the two other points and then back to (0,0).

The line integral around the loop will be 0, so the remaining value must "balance" the integral to 0. For form A, it is -4000, for Form B, it is $-30{,}375$.

11. (5 pts ec) Our final is May 6 from 7 pm to 9 pm in room ss-229