**Practice Problems, Test 3/Final, MAT267**

1. Evaluate this integral using polar notation:

$$\int_{-4}^{-3}\int_{0}^{\sqrt{16-x^2}}(x^2+y^2)\,dy\ dx+\int_{-3}^{3}\int_{\sqrt{9-x^2}}^{\sqrt{16-x^2}}(x^2+y^2)\,dy\ dx+\int_{3}^{4}\int_{0}^{\sqrt{16-x^2}}(x^2+y^2)\,dy\ dx.$$

2. Suppose region $E$ is between two hemispheres of radius 2 and radius 5 above the *xy*-plane. Set us and evaluate $\iiint_E\ x^2+y^2+z^2\ dV$.

3. Set up an integral and find the volume contained in the solid bounded by the *xy*-plane, the plane $z=x$, the paraboloid $x=9-y^2$ such that *x* is positive.

4. Find the volume within the region bounded by $z=x^2+y^2$ and $z=32-x^2-y^2$.

5. Find $\iiint_E\ dV$ where $E$ is the tetrahedron with vertices (0,0,0), (2,0,0), (0,3,0) and (0,0,6).

6. Convert the rectangular coordinate (2, –2, 5) to $(\rho,\theta,\varphi)$.

7. A solid is bounded below by a circular cone (vertex at the origin) and above by a sphere (center at the origin) such that (2,1,5) lies on the rim where the cone and sphere intersect. (This solid is called a *spherical wedge*. It looks like an ice-cream cone). Find its volume.

8. A particle follows a straight-line path from (1,2) to (5,7) within the vector field $F(x,y)=\langle xy,y^2\rangle$. Find the work. (That is, find $\int_C\ F\cdot dr$ where *C* is the path of the particle.)

9. Show that $F(x,y)=\langle 6x+5y,5x+4\rangle$ is conservative, then find $f(x,y)$ such that $\nabla f=F$.

10. Find $\int_C\ F\cdot dr$ where $F(x,y)=\langle 4xy^3,6x^2y^2\rangle$ and *C* is a sequence of straight lines from (0,0) to (1,3) to (4,7) to (9,5) to (2,1).

11. Find $\int_C\ F\cdot dr$ where $F(x,y)=\langle 3y,-2x\rangle$ and *C* is the path starting at (0,0) to (4,0) to (4,4) back to (0,0).

12. Find $\int_C\ F\cdot dr$ where $F(x,y)=\langle 10y,12x\rangle$ and *C* is a circle of radius 4 centered at the origin traced clockwise.

13. Find $\int_C\ F\cdot dr$ where $F(x,y)=\langle \sin y\,,x\cos y\rangle$ and *C* is an ellipse centered at (5,4) with minor axis 7 and major axis 43, traversed clockwise on a Tuesday with Pink Floyd's "Animals" playing in the person's earbuds.

14. Find the work done by $F(x,y)=\langle z,-z,x^2-y^2\rangle$ along the path from (2,0,0) to (0,4,0) to (0,0,8) back to (2,0,0).

## Answers and discussion.

1. The polar integral is $\int_0^\pi \int_3^4 r^3\ dr\ d\theta$. It evaluates to $\frac{175}{4}\pi$.
2. This is best done in spherical coordinates. We have $\int_0^{2\pi}\int_0^{\pi/2}\int_2^5 \rho^4 \sin\varphi\ d\rho\ d\varphi\ d\theta$. It evaluates to $\frac{6186}{5}\pi$.
3. The integral is $\int_{-3}^{3}\int_0^{9-y^2}\int_0^x dz\ dx\ dy$. It evaluates to $\frac{648}{5}$.
4. The two surfaces intersect at $x^2+y^2=16$, a circle of radius 4. Thus, it's best to use cylindrical coordinates. The integral is $\int_0^{2\pi}\int_0^4\int_{r^2}^{32-r^2} dz\ r\,dr\ d\theta$. It evaluates to $256\pi$.
5. Let's use a *dz dy dz* ordering. The sloping face is a plane, $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$, or $3x+2y+z=6$. Solving for *z*, we get $z=6-3x-2y$. Thus, the bounds for z are $0\le z\le 6-3x-2y$. Now looking at the "footprint" region in the xy-plane, and choosing to integrate with respect to y, the line connecting (2,0) and (0,3) is $y=3-\frac{3}{2}x$, so the y-bounds are $0\le y\le 3-\frac{3}{2}x$. The x-bounds are $0\le x\le 2$, and the integral is $\int_0^2\int_0^{3-\frac{3}{2}x}\int_0^{6-3x-2y} dz\ dy\ dx$. It evaluates to 6.
6. The radius is $\rho=\sqrt{2^2+(-2)^2+5^2}=\sqrt{33}$. The x and y coordinates lie in quadrant 4, at an angle of 45 degrees sloping downward, or $-\frac{\pi}{4}$. However, we use the positive equivalent, $\frac{7\pi}{4}$. Angle $\varphi=\tan^1\frac{2\sqrt{2}}{5}\approx 0.5148$. Thus, (2, –2, 5) is equivalent to $\left(\sqrt{33},\frac{7\pi}{4},0.5148\right)$.
7. The good news is that in spherical coordinates, all bounds are constants. For $\rho$, we have $\sqrt{2^2+1^2+5^2}=\sqrt{30}$, so $0\le\rho\le\sqrt{30}$. Since the region sweeps entirely around the z-axis, we have $0\le\theta\le 2\pi$. To find bounds for $\varphi$, note that the point (2,1,5) forms one corner of a right-triangle with (0,0,0) and (0,0,5) as the other corners, so that $\cos\varphi=\frac{5}{\sqrt{30}}$. Thus, the bounds for $\varphi$ are $0\le\varphi\le\cos^{-1}\frac{5}{\sqrt{30}}$. The integral is $\int_0^{2\pi}\int_0^{\cos^{-1}\frac{5}{\sqrt{30}}}\int_0^{\sqrt{30}}\rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta$. The inner integral gives $\int_0^{\sqrt{30}}\rho^2\,d\rho=\frac{1}{3}\left(\sqrt{30}\right)^3$. This is a constant so we move it to the front. Next, we integrate with respect to $\varphi$: $\int_0^{\cos^{-1}\frac{5}{\sqrt{30}}}\sin\varphi\ d\varphi\ =[-\cos\varphi]_0^{\cos^{-1}\frac{5}{\sqrt{30}}}=-\cos\left(\cos^{-1}\frac{5}{\sqrt{30}}\right)-(-\cos 0)=-\frac{5}{\sqrt{30}}+1$. Lastly, $\int_0^{2\pi}d\theta=2\pi$. Thus, the volume is the product of these three constants: $\frac{1}{3}\left(\sqrt{30}\right)^3\left(1-\frac{5}{\sqrt{30}}\right)2\pi$.
8. The vector field is not conservative, so we must parameterize the path. We get $r(t)=\langle 1+4t,2+5t\rangle$, for $0\le t\le 1$. Thus, $dr=\langle 4,5\rangle$. Furthermore, substituting to get $F$ in terms of $t$ gives the following: $F(x(t),y(t))=\langle(1+4t)(2+5t),(2+5t)^2\rangle=\langle 20t^2+13t+2,25t^2+20t+4\rangle$. Thus, $F\cdot dr=4(20t^2+13t+2)+5(25t^2+20t+4)=205t^2+152t+28$. This is integrated from 0 to 1, and you get $\frac{517}{3}$.
9. We have $M_y=5=N_x$, so it's conservative. We need $f(x,y)$ such that $f_x=M$ and $f_y=N$. So we integrate: $\int M\,dx=\int(6x+5y)dx=3x^2+5xy$ and $\int N\,dy=\int(5x+4)dy=5xy+4y$. Thus, $f(x,y)=3x^2+5xy+4y$, which can be easily checked to show that $\nabla f=F$.
10. Always check to see if the field is conservative. If it is, you don't need to parameterize paths! In this case, we have $M_y=12xy^2=N_x$, so F is conservative. We find its potential function. Using a technique like in #9, we find that $f(x,y)=2x^2y^3$. Thus, $\int_C\ F\cdot dr=2x^2y^3\big|_{(0,0)}^{(2,1)}=2(2)^2(1)^3=8$. You only need to evaluate between the endpoints.

11. The vector field is not conservative, but the path is a closed loop, so we use Green's Theorem: $\iint_R (N_x - M_y)\ dA$. The integrand is –5, so we have $-5\iint_R\ dA$, where the double integral is just the area over the region, which is a triangle with base 4 and height 4, so $-5\iint_R\ dA = -5(8) = -40$.
12. Like #11, the field F is not conservative but the path is a loop. We find that $N_x - M_y = 2$, so using Green's Theorem, and recognizing that the region is a circle with radius 4, the line integral is $2(\pi(4)^2) = 32\pi \ldots$. But wait! The path was traversed clockwise. To use Green's Theorem, we must traverse counterclockwise, so the actual result is $-32\pi$.
13. The vector field is conservative and the path is a loop, so the answer is 0.
14. We need two things: $\nabla \times F$, which is $\langle 1-2y, 1-2x, 0\rangle$, and a normal vector $n$ to this surface. The plane passing through the three given points is $\frac{x}{2} + \frac{y}{4} + \frac{z}{8} = 1$, or $4x + 2y + z = 8$, and in this form of a plane's equation, the normal is just the coefficients, so $n = \langle 4,2,1\rangle$. Thus, we calculate $\iint_R (\nabla \times F) \cdot n\ dS = \int_0^2 \int_0^{4-2x} (6 - 4x - 8y)\ dy\ dx = -\frac{88}{3}$. Note that the bounds of the double integral refer to the "footprint" made by this surface over the *xy*-plane, which in this case is just a triangle.

As usual, if you see an error, please let me know, surgent@asu.edu