Double Integrals using Polar Coordinates

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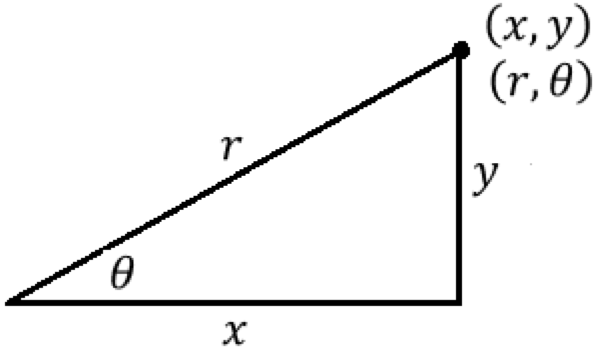
Regions that are formed by circles are better described using **polar coordinates**.

If (r, θ) represents a point in the plane, then r is the distance from the point to the origin, and θ represents the angle that a ray from the origin to the point makes with the positive *x*-axis.

The usual conversion formulas between rectangular (x, y) coordinates to polar (r, θ) coordinates are:

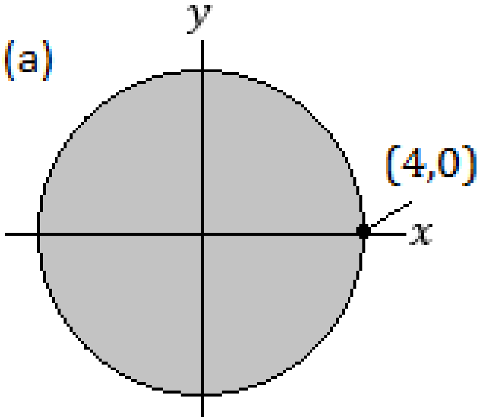
(x, y) to (r, θ): $\begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$ (r, θ) to (x, y): $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$



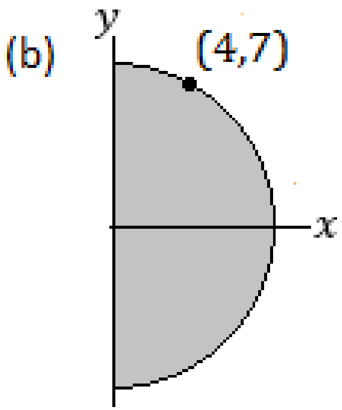
Circular regions in the *xy*-plane can be described using polar coordinates where $a \le r \le b$ and $c \le \theta \le d$, and a, b, c and d are constants.

Such regions are called **polar rectangles**.

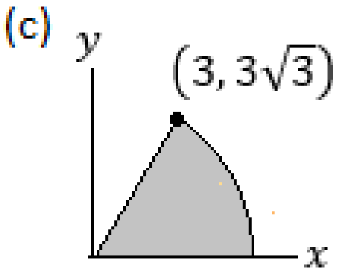
Example 1: Describe the following regions using polar coordinates.



$0 \le r \le 4,$
$0 \le \theta \le 2\pi.$

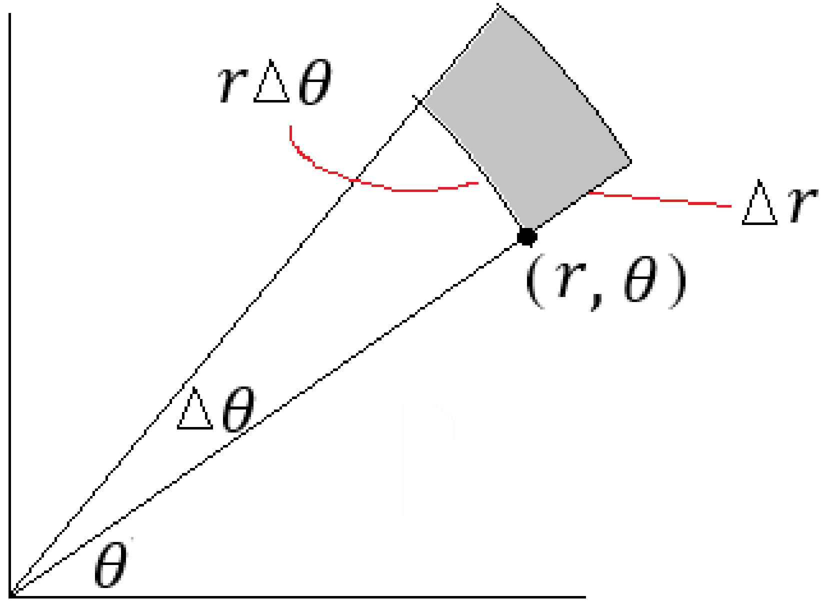


$0 \le r \le \sqrt{65},$
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$0 \le r \le 6,$
$0 \le \theta \le \frac{\pi}{3}$

The **polar integral area element**, also known as the **Jacobian**.



Let (r, θ) be a point in R^2 described in polar coordinates.

Extend r slightly, by Δr units.

Allow the angle θ to increase, by $\Delta\theta$ units.

The length of an arc of a circle of radius r subtended by θ radians is $S = r\theta$. In this case, the subtending angle is $\Delta\theta$.

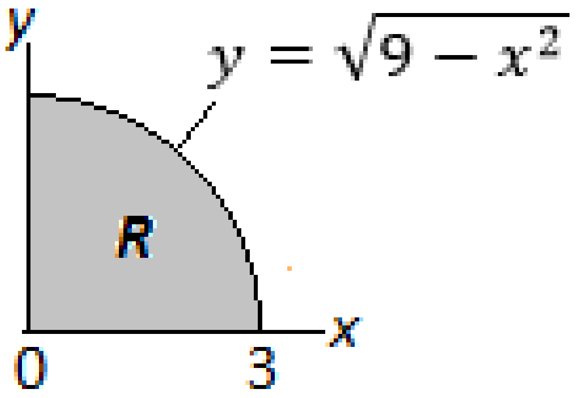
The length of this arc is $r\Delta\theta$.

The area of this small region is $(r\Delta\theta)(\Delta r)$.

On small scales, use differentials. The Jacobian of the polar integral is $\boldsymbol{r\ dr\ d\theta}$.

Example 2: Evaluate

$$\int_0^3 \int_0^{\sqrt{9-x^2}} xy\, dy\, dx\,.$$



Solution: The region of integration is a quarter circle in the first quadrant, center at the origin, radius 3.

The bounds of integration are $0 \le r \le 3$ and $0 \le \theta \le \frac{\pi}{2}$. Furthermore, we substitute $x = r\cos\theta$ and $y = r\sin\theta$, and exchange $dy\, dx$ with $r\, dr\, d\theta$:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} xy\, dy\, dx = \int_0^{\pi/2} \int_0^3 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta = \int_0^{\pi/2} \int_0^3 r^3 \cos\theta \sin\theta\ dr\, d\theta\,.$$

The inside integral is evaluated first:

$$\int_0^3 r^3 \cos\theta \sin\theta\ dr = \cos\theta \sin\theta \left[\frac{1}{4} r^4\right]_0^3 = \frac{81}{4}\cos\theta \sin\theta\,.$$

(Ex. 2 continued) This is integrated with respect to θ, using *u*-*du* substitution, with $u = \sin\theta$ and $du = \cos\theta$:

$$\frac{81}{4}\int_0^{\pi/2} \cos\theta \sin\theta \; d\theta = \left[\frac{81}{8}\sin^2\theta\right]_0^{\pi/2}$$

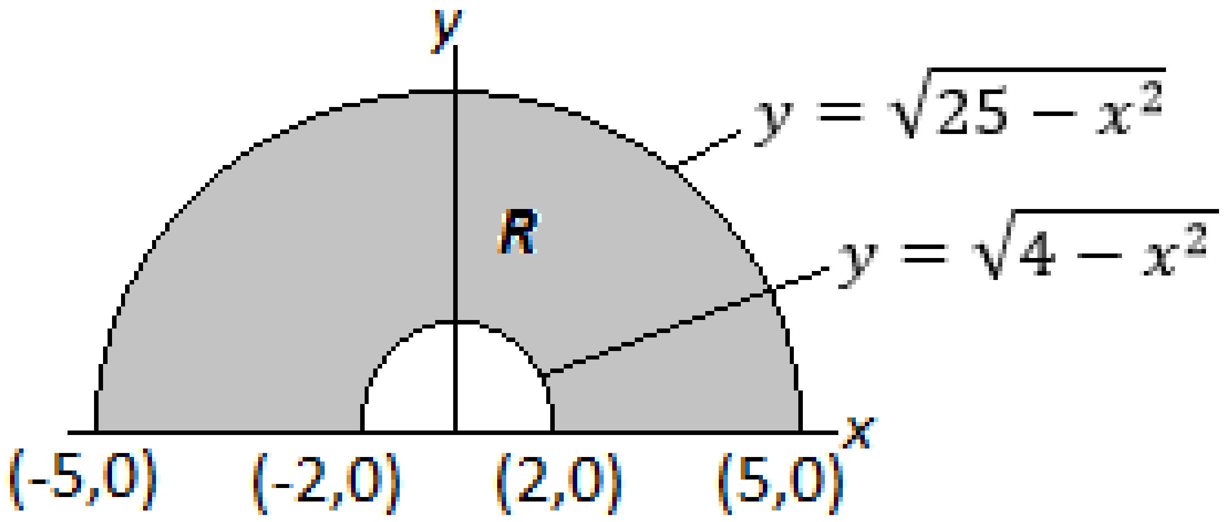
$$= \frac{81}{8}[1^2 - 0]$$

$$= \frac{81}{8}\ .$$

Example 3: Evaluate

$$\int_{-5}^{-2}\int_{0}^{\sqrt{25-x^2}} x^2\,dy\,dx + \int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{25-x^2}} x^2\,dy\,dx + \int_{2}^{5}\int_{0}^{\sqrt{25-x^2}} x^2\,dy\,dx\,.$$

Solution: The region of integration as suggested by the bounds in the three integrals is shown below



In polar coordinates, the region is $2 \leq r \leq 5$ and $0 \leq \theta \leq \pi$.

Replace the integrand x^2 with $(r \cos\theta)^2 = r^2 \cos^2\theta$, and the area element $dy\, dx$ with $r\, dr\, d\theta$.

The three double integrals in rectangular coordinates are equivalent to one double integral in polar coordinates, with constant bounds:

$$\int_0^{\pi} \int_2^5 r^2 \cos^2\theta \; r\, dr\, d\theta\,, \qquad \text{which simplifies to} \qquad \int_0^{\pi} \int_2^5 r^3 \cos^2\theta \; dr\, d\theta\,.$$

The inside integral with respect to r is evaluated first:

$$\int_2^5 r^3 \cos^2\theta \; dr = \cos^2\theta \left[\frac{1}{4} r^4\right]_2^5 = \frac{1}{4}\cos^2\theta\,[(5)^4 - (2)^4] = \frac{609}{4}\cos^2\theta\,.$$

This expression is next integrated with respect to θ.

To antidifferentiate $\cos^2\theta$, use the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$:

$$\int_0^{\pi} \frac{609}{4}\cos^2\theta\, d\theta = \frac{609}{4}\int_0^{\pi}\left(\frac{1}{2}(1+\cos 2\theta)\right)d\theta$$

$$= \frac{609}{8}\int_0^{\pi} 1+\cos 2\theta \;\; d\theta$$

$$= \frac{609}{8}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi}$$

$$= \frac{609}{8}\left[\left(\pi + \frac{1}{2}\sin 2(\pi)\right) - \left(0 + \frac{1}{2}\sin 2(0)\right)\right]$$

$$= \frac{609}{8}\pi\,.$$

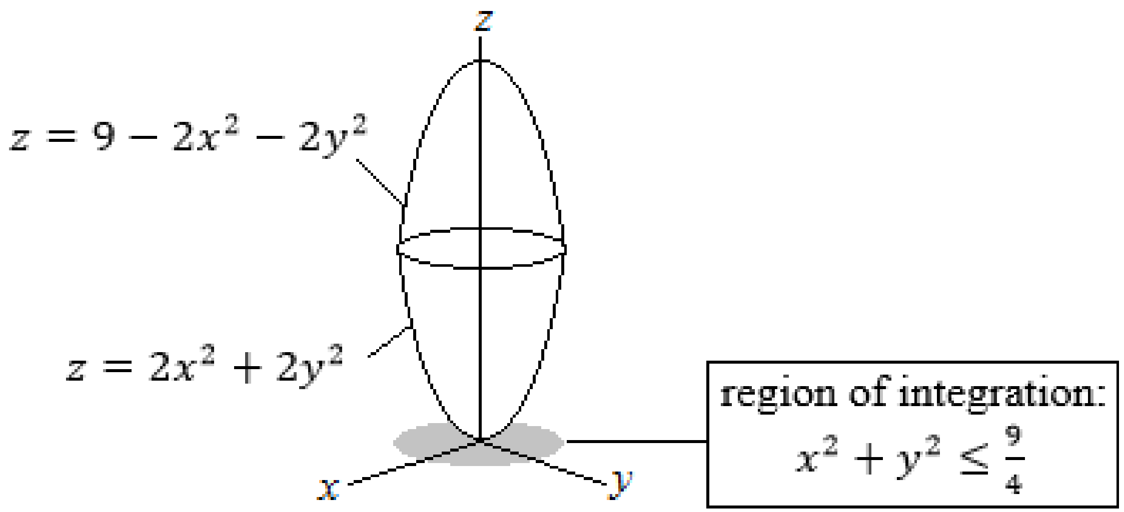
Example 4: Find the volume of the solid bounded by $z = 2x^2 + 2y^2$ and $z = 9 - 2x^2 - 2y^2$.

Solution: Set the two functions equal and simplify:

$$2x^2 + 2y^2 = 9 - 2x^2 - 2y^2$$

$$4x^2 + 4y^2 = 9$$

$$x^2 + y^2 = \frac{9}{4}\ .$$



The region of integration is the disk $x^2 + y^2 \leq \frac{9}{4}$, which can be described in polar coordinates as $0 \leq r \leq \frac{3}{2}$ and $0 \leq \theta \leq 2\pi$.

To the right is a sketch of the solid along with the region of integration.

The integrand is the "top" boundary ($z = 9 - 2x^2 - 2y^2$) subtracted by the "bottom" boundary ($z = 2x^2 + 2y^2$). This is

$$9 - 2x^2 - 2y^2 - (2x^2 + 2y^2) = 9 - 4x^2 - 4y^2$$

$$= 9 - 4(x^2 + y^2)$$

$$= 9 - 4r^2.$$

The volume is found by evaluating

$$\int_0^{2\pi} \int_0^{3/2} (9 - 4r^2)\, r\, dr\, d\theta\,, \qquad \text{or simplified as:} \qquad \int_0^{2\pi} \int_0^{3/2} (9r - 4r^3)\, dr\, d\theta\,.$$

For the inner integral, we have

$$\int_0^{3/2} (9r - 4r^3)\, dr = \left[\frac{9}{2}r^2 - r^4\right]_0^{3/2}$$

$$= \frac{9}{2}\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^4 - 0$$

$$= \frac{243}{16}.$$

Finally, the volume is

$$\int_0^{2\pi} \frac{243}{16}\, d\theta = \frac{243}{16}\int_0^{2\pi} d\theta$$

$$= \frac{243}{16}(2\pi)$$

$$= \frac{243\pi}{8}\ \text{units}^3.$$