Multivariable Functions

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A function in R^3 has two independent variables, and a third variable dependent on the first two.

If x and y represent the independent variables, and z the dependent variable, a **function in two variables** can be written $z = f(x, y)$.

Depending on the situation, we can let y be the dependent variable, so that $y = f(x, z)$, or let x be the dependent variable, so that $x = f(y, z)$.

The **domain** of an n-variable function is the set of ordered n-tuples in R^n for which the function is defined.

The **range** is the set of values in R^1 for which the dependent variable can assume.

The visual representation of the set of points (ordered n-tuples) for which a function is defined is called a **graph**. In R^3, the graph is often called a **surface**.

Example 1: Given $z = f(x, y) = \frac{1}{x} + 2y$. Find $f\left(\frac{1}{3}, 4\right)$ and the domain of f.

Solution: We have

$$f\left(\frac{1}{3}, 4\right) = \frac{1}{1/3} + 2(4) = 3 + 8 = 11.$$

This is an ordered triple $\left(\frac{1}{3}, 4, 11\right)$ on the graph of f. Note that since x is in the denominator, we must have $x \neq 0$. Thus, the domain is the set of x and y values for which $x \neq 0$. Using set-builder notation, we can write this as

$$\text{Dom } f = \{(x, y) |\ x \in R \text{ and } y \in R \text{ such that } x \neq 0\}.$$

Example 2: Find the domain and range of $z = g(x, y) = \sqrt{81 - x^2 - y^2}$.

Solution: The expression inside the radical must be non-negative. Thus, we have

$$81 - x^2 - y^2 \geq 0.$$

Rearranging the terms, the domain of g is $\{(x, y) \mid x^2 + y^2 \leq 81\}$.

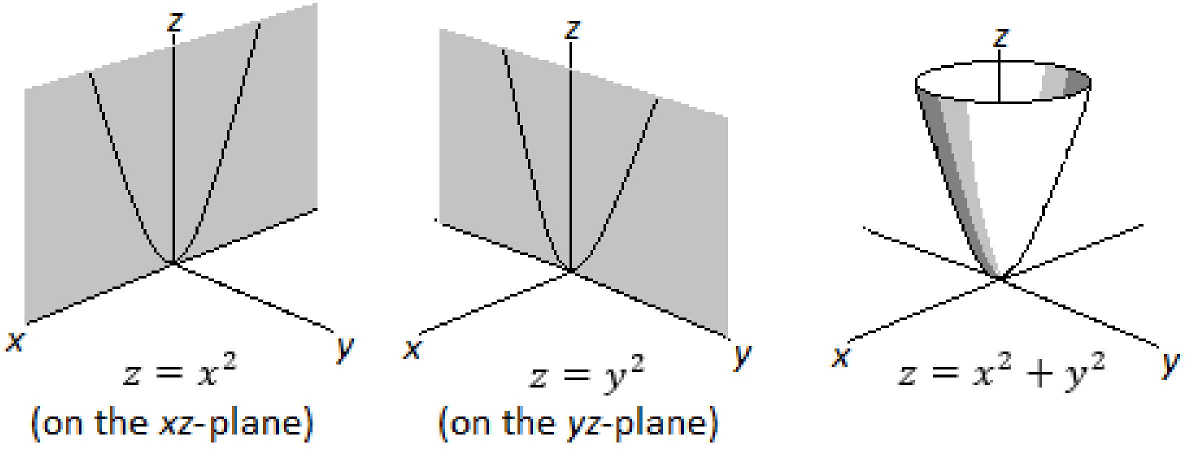
The surface of g is a hemisphere of radius 9, and its domain is a filled-in circle of radius 9, centered at (0,0) on the *xy*-plane. The range of g is $\{z \mid 0 \leq z \leq 9\}$.

Example 3: Sketch $z = x^2 + y^2$.

Solution: When $x = 0$, we have $z = y^2$, which is a parabola opening in the positive z direction on the yz-plane.

Similarly, when $y = 0$, we have another parabola $z = x^2$ opening in the positive z direction on the xz-plane.

Together, the two parabola traces suggest that the surface of the function $z = x^2 + y^2$ is a parabolic bowl, or **paraboloid**.

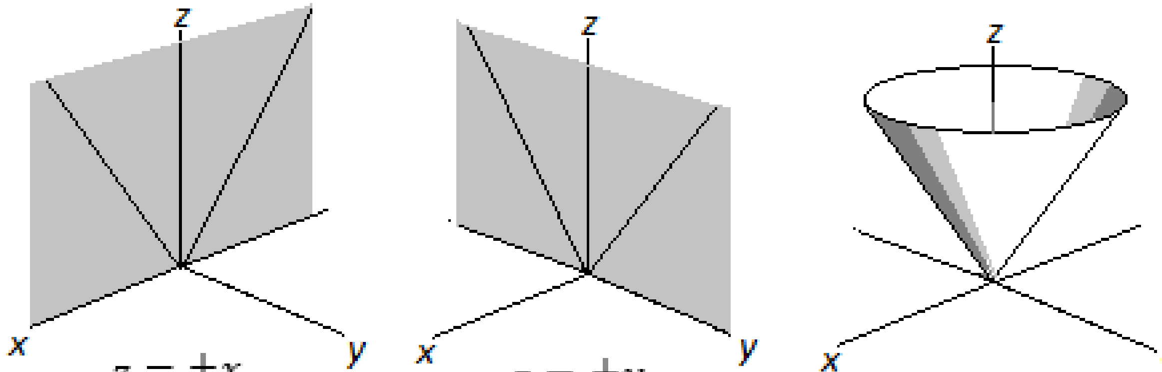


Note that this paraboloid has a vertex at (0,0,0). If positive z is considered "up", then we say this paraboloid opens upward. The domain is $\{(x, y)|x \in R, y \in R\}$, and the range is $\{z|z \geq 0\}$.

Example 4: Describe the surface $z = \sqrt{x^2 + y^2}$.

Solution: First, note that $x^2 + y^2 \geq 0$ for all x and y, so that the domain is $\{(x, y)|x \in R, y \in R\}$. Note also that the radical results in non-negative values for z, so that the range is $\{z|z \geq 0\}$.

Let $y = 0$, so that means $z = \sqrt{x^2 + 0} = \pm x$. Similarly, when $x = 0$, we have $z = \sqrt{0 + y^2} = \pm y$. These are lines that form a "V" shape in their respective planes. The cross sections parallel to the *xy*-plane are circles, and together, these facts suggest that $z = \sqrt{x^2 + y^2}$ is a **cone**.



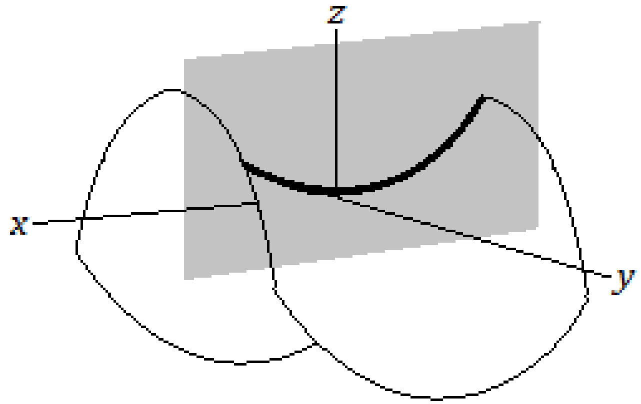
$z = \pm x$
(on the *xz*-plane)

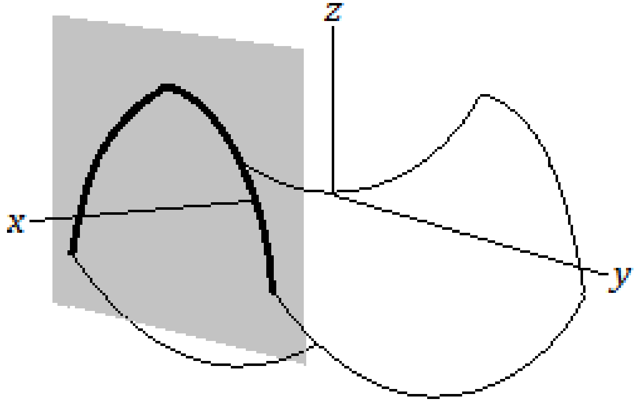
$z = \pm y$
(on the *yz*-plane)

$z = \sqrt{x^2 + y^2}$

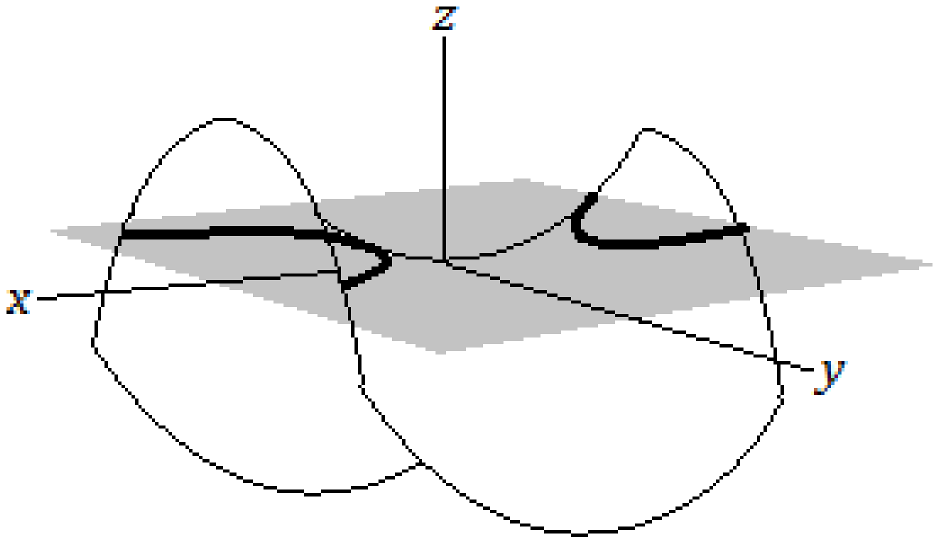
Example 5: Describe the surface $z = x^2 - y^2$.

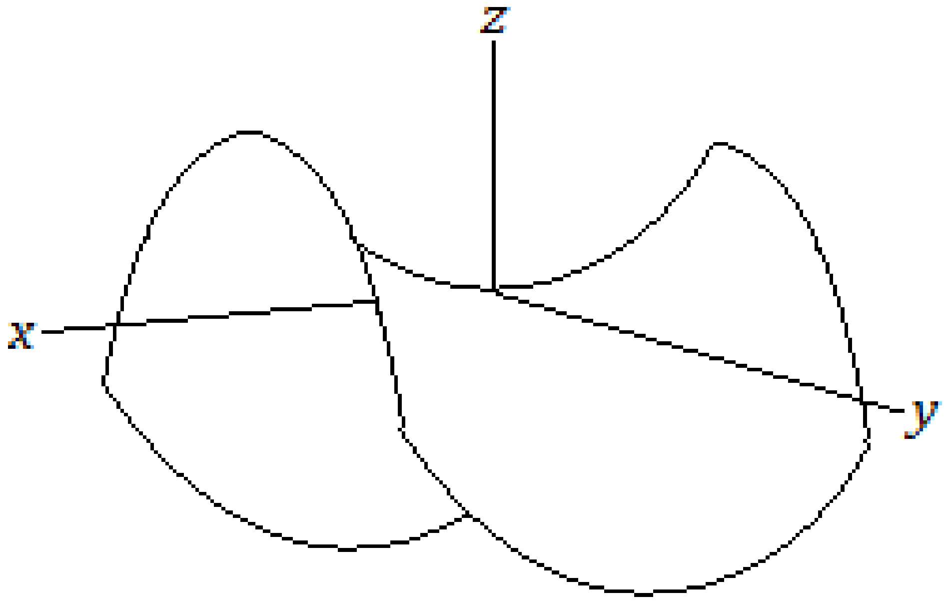
Solution: When $y = 0$, the surface's trace on the *xz*-plane is $y = x^2$, a parabola that opens in the positive *z* direction. When $x = 0$, the trace on the *yz*-plane is $z = -y^2$, a parabola that opens in the negative *z* direction.





When a plane parallel to the *xz*-plane intersect the surface $z = x^2 - y^2$, it forms a parabola that opens up. When a plane parallel to the *yz*-plane intersects the surface, it forms a parabola that opens down.





The surface is called a **hyperbolic paraboloid**. It is shaped like a saddle and is informally called a saddle. The origin in this case is the saddle point.

Example 6: Describe the surface $x^2 + y^2 - z^2 = 1$.

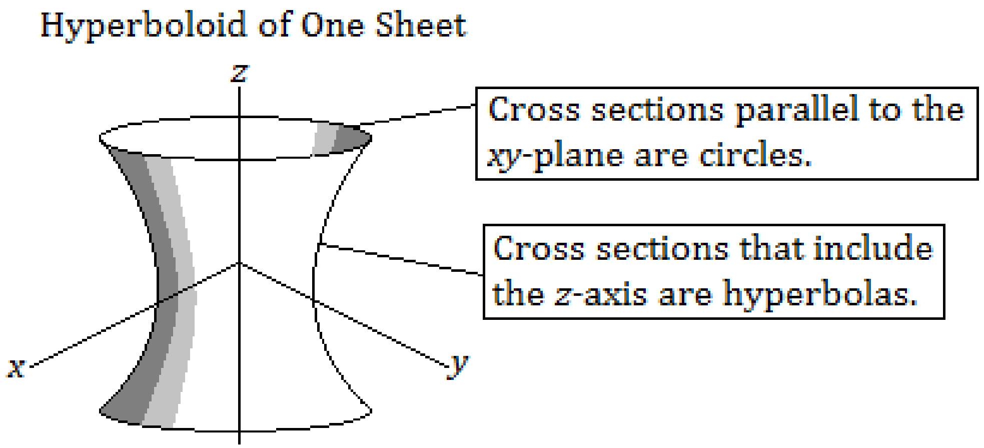
Solution: We can infer the surface's appearance by setting each variable to 0, one at a time.

When $x = 0$, we have $y^2 - z^2 = 1$, which is a hyperbola in the *yz*-plane where the two halves open in the positive and negative *y*-directions.

When $y = 0$, we have $x^2 - z^2 = 1$, which is a hyperbola in the *xz*-plane where the two halves open in the positive and negative *x*-directions.

When $z = 0$, we have $x^2 + y^2 = 1$, which is a circle of radius 1 in the *xy*-plane, centered at the origin.

The resulting shape is called a *hyperboloid of one sheet*. Note that it does not intersect the *z*-axis (the *z*-axis is the axis of symmetry of this surface).



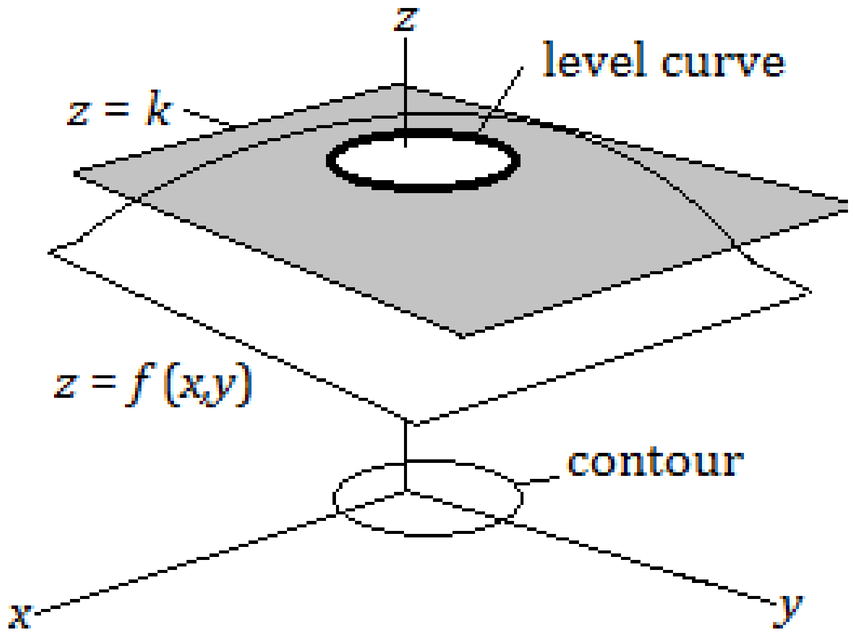
Level Curves and Contour Maps

The intersection of $z = f(x, y)$ with $z = k$ forms a **level curve** on the surface of f.

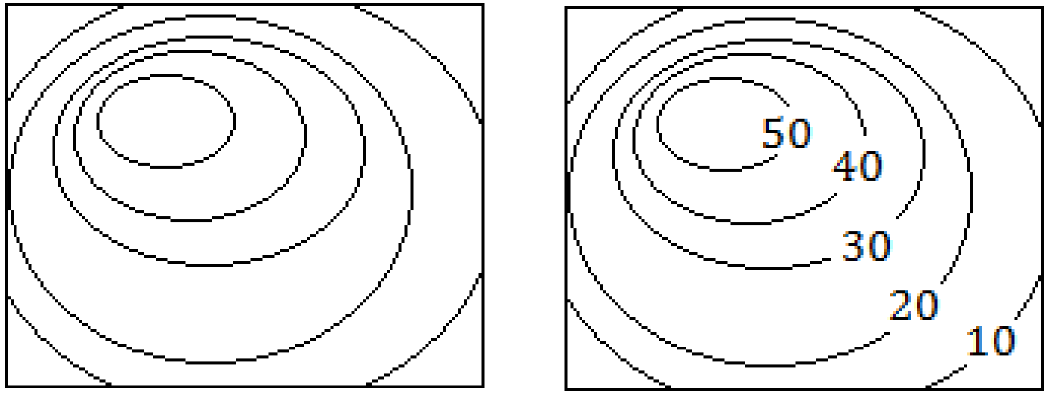
It is a curve where x and y can vary, but z does not change.

Imagine standing on a hill and taking a step such that you neither go uphill nor downhill.

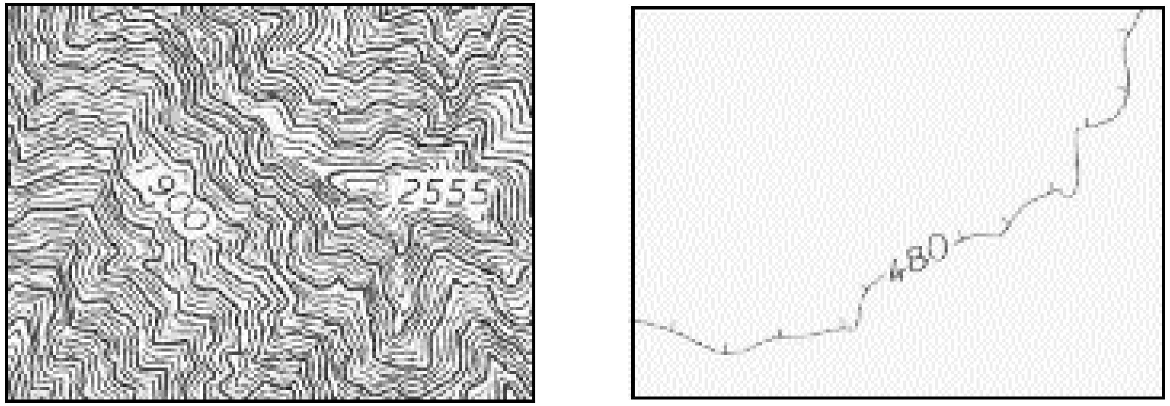
If you do this repeatedly, you will (theoretically) walk along a path that is level, and end at the same point from which you started.



The contours are marked by the *z*-values for reference. For complicated surfaces, the *z*-values may differ by some set value, *e.g*. by multiples of 10. Too many contours, and the map is cluttered, while too few contours, important details may be lost.

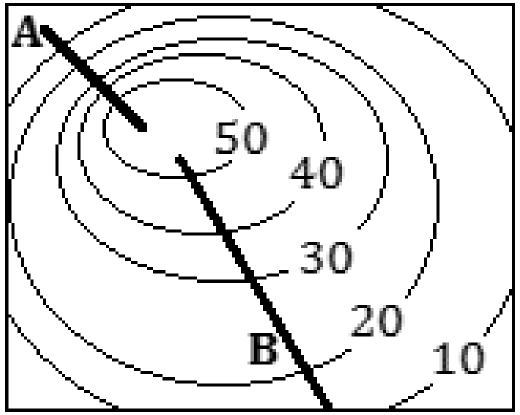


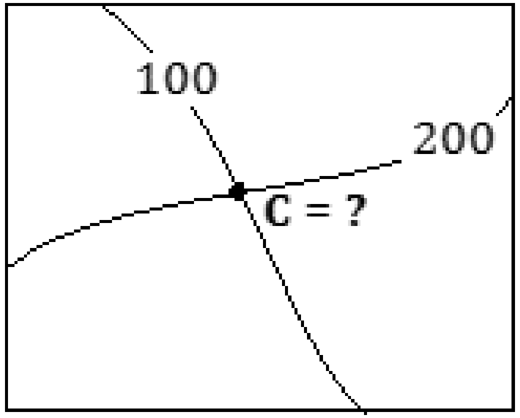
Left: This is a vague contour map. There are no *z*-values. Is this a hill or a basin?
Right: With *z*-values now in place, we see that this represents a hill.

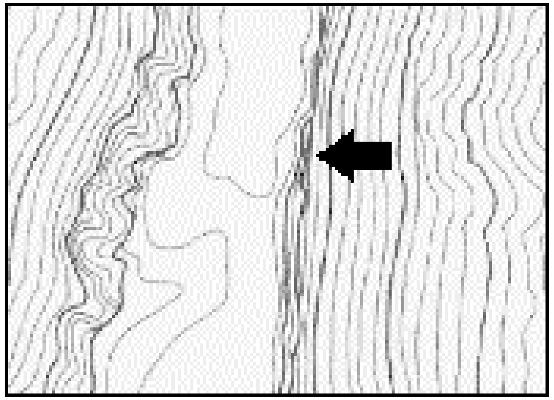


Left: The map is too cluttered. **Right:** Now it's too vague.

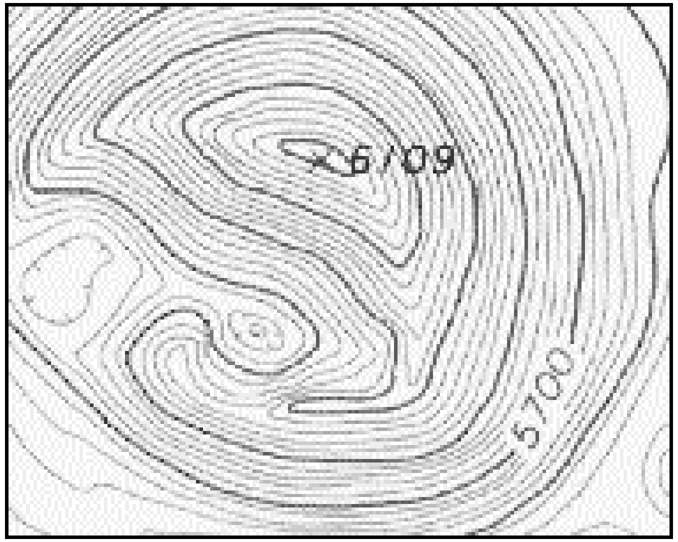
Contour lines for different z-values never touch. This would violate the vertical-line rule for functions in R^3. Contours that are close together indicate a large change in z, or “steepness”, while contours spaced apart indicate flatter terrain. Some topographical maps may show what appears to be contours that “touch”. These represent cliffs.

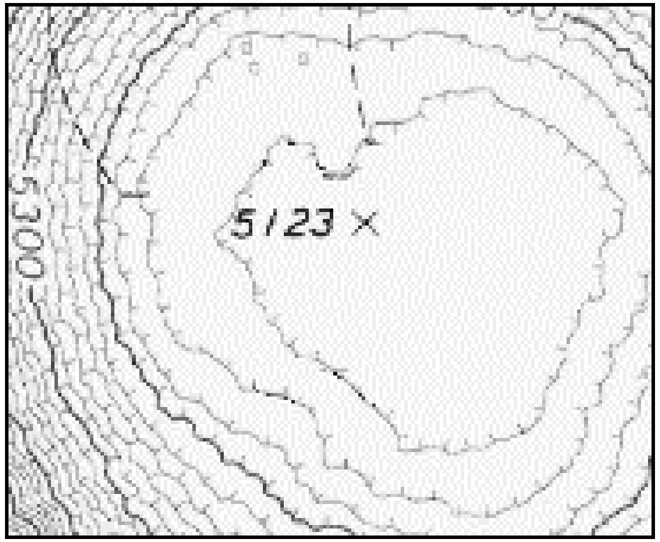






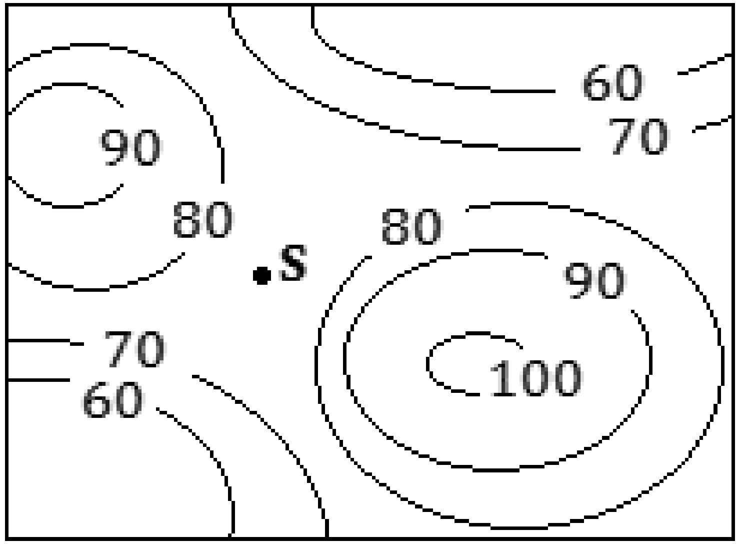
Contours can form closed loops and nest (and or may be nested within) other contours. Hills are found where the nested contours increase in values of *z*. Basins are found where the nested contours decrease in values of *z*.

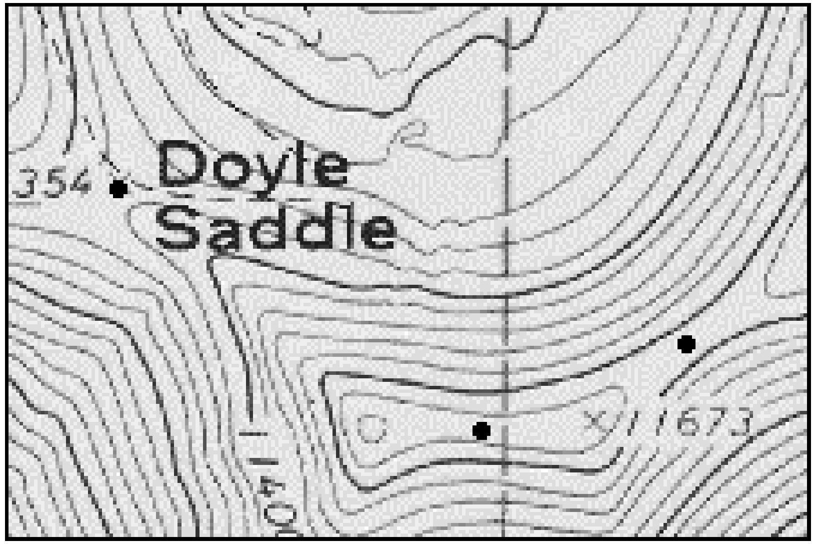




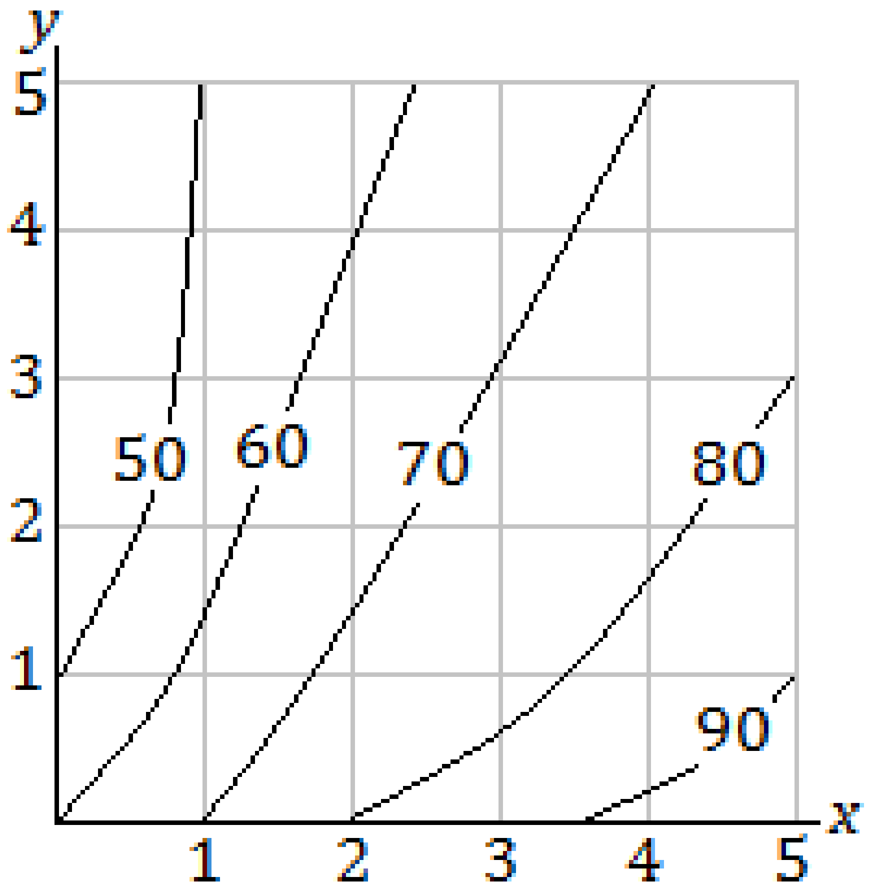
Saddle points are locations where the point may be a minimum along some paths crossing through the point, and a maximum for other paths passing through the point.

On most contour maps, a saddle is usually inferred where a contour "pinches" in, then widens again.





Example 7: Given the contour map below for $z = f(x, y)$ where $0 \leq x \leq 5$ and $0 \leq y \leq 5$. Find the following: (a) $f(2,4)$; (b) $f(3,2)$; (c) $f(3,4)$; (d) $f(2, y) = 70$.

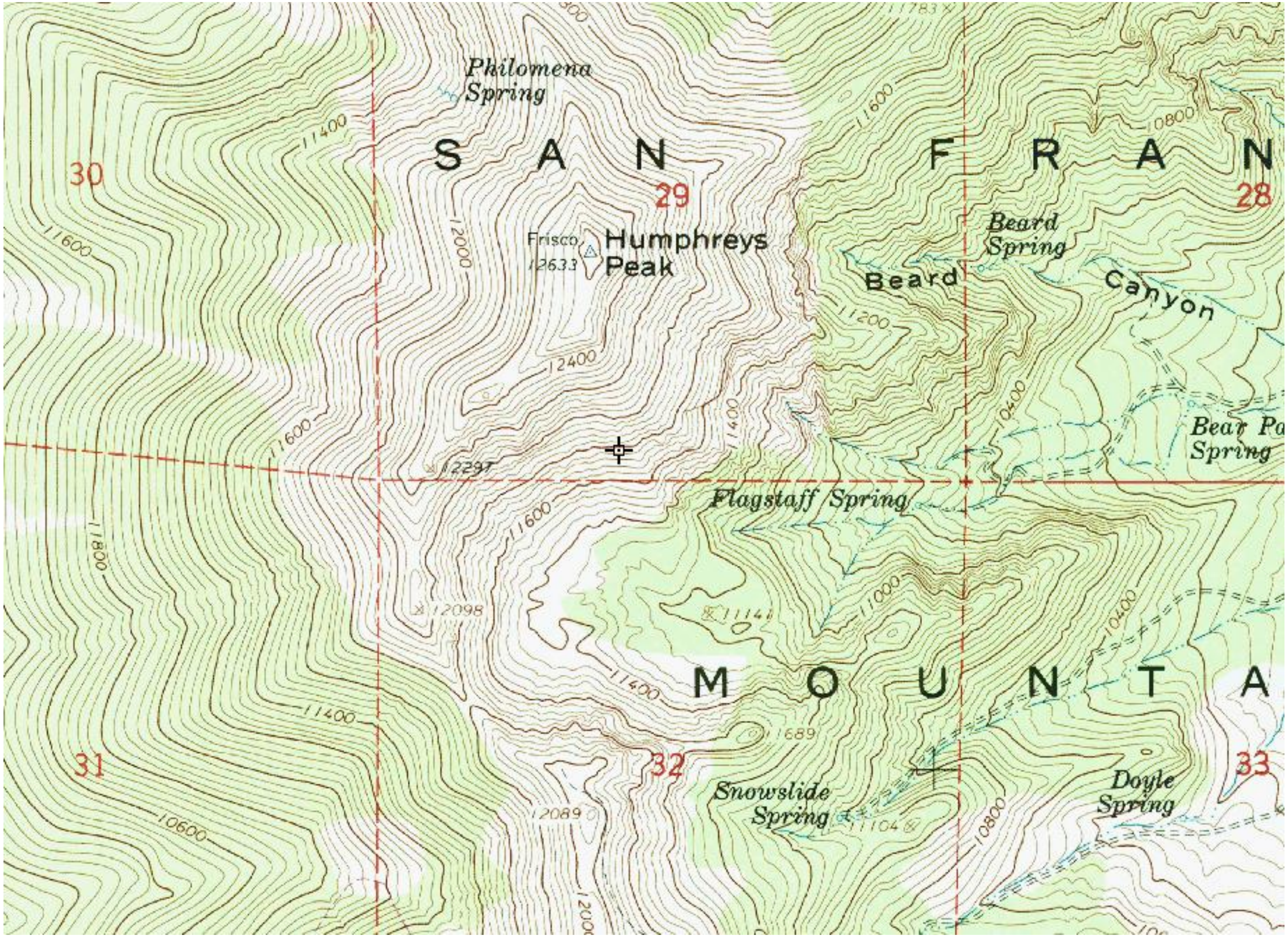


a) $f(2,4) = 60$.

b) $f(3,2) \approx 73$, for example. However, any z value such that $70 < z < 80$ would be a plausible answer

c) $f(3,4) \approx 67$, although any z value such that $60 < z < 70$ is a plausible answer

d) $f(2, 1.5) = 70$



Philomena Spring
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Humphreys Peak
Beard Spring
Beard Canyon
Bear Pa
Spring
Flagstaff Spring
Snowslide Spring
Doyle Spring
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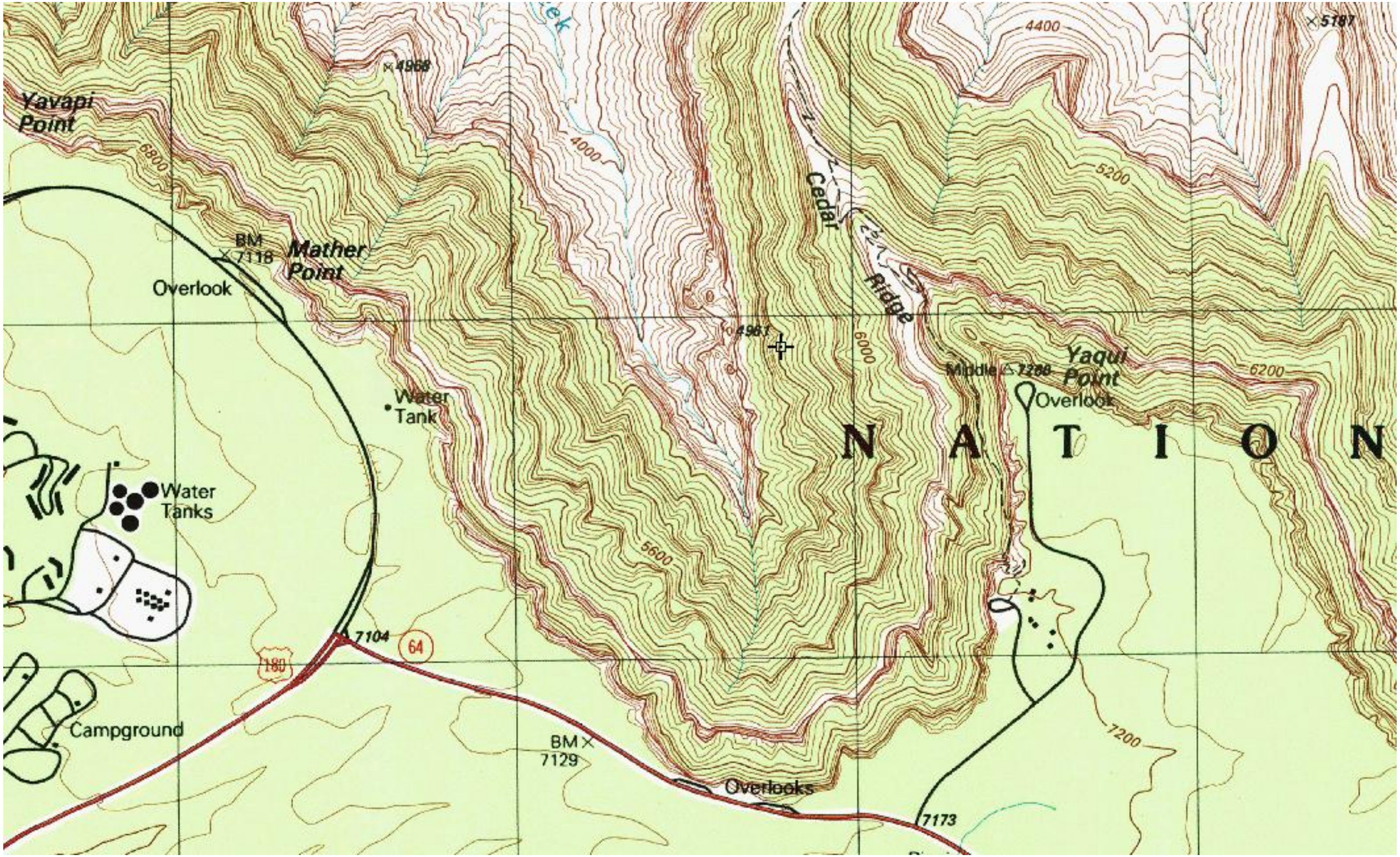
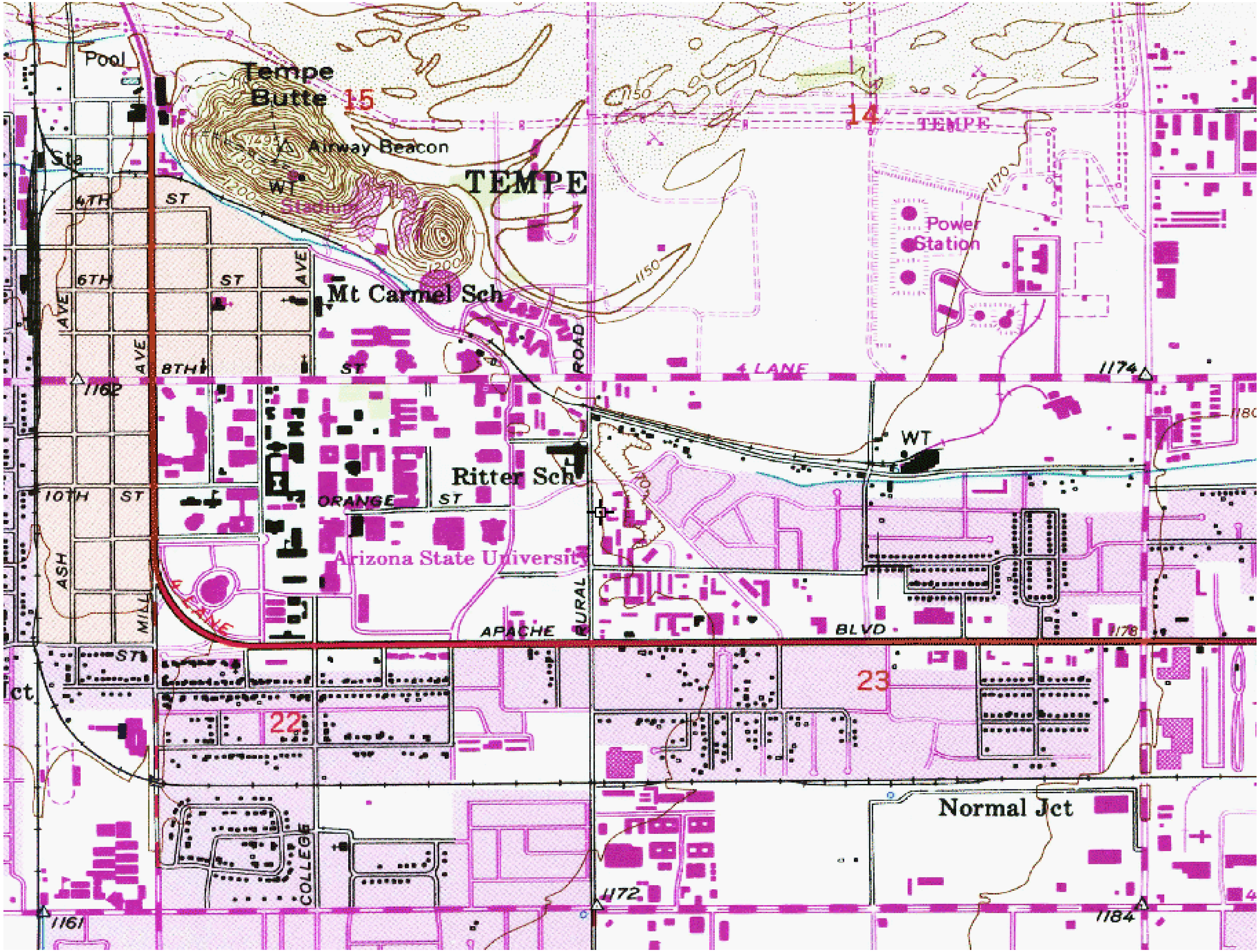


Yavapi Point
6800
BM 7118
Mather Point
Overlook
4968
4000
4400
5187
Cedar Ridge
5200
4961
6000
Yaqui Point
Middle
Overlook
6200
N A T I O N
Water Tank
Water Tanks
5600
7104
64
180
Campground
BM 7129
Overlooks
7173
7200



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TEMPE
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10TH ST
ORANGE ST
Arizona State University
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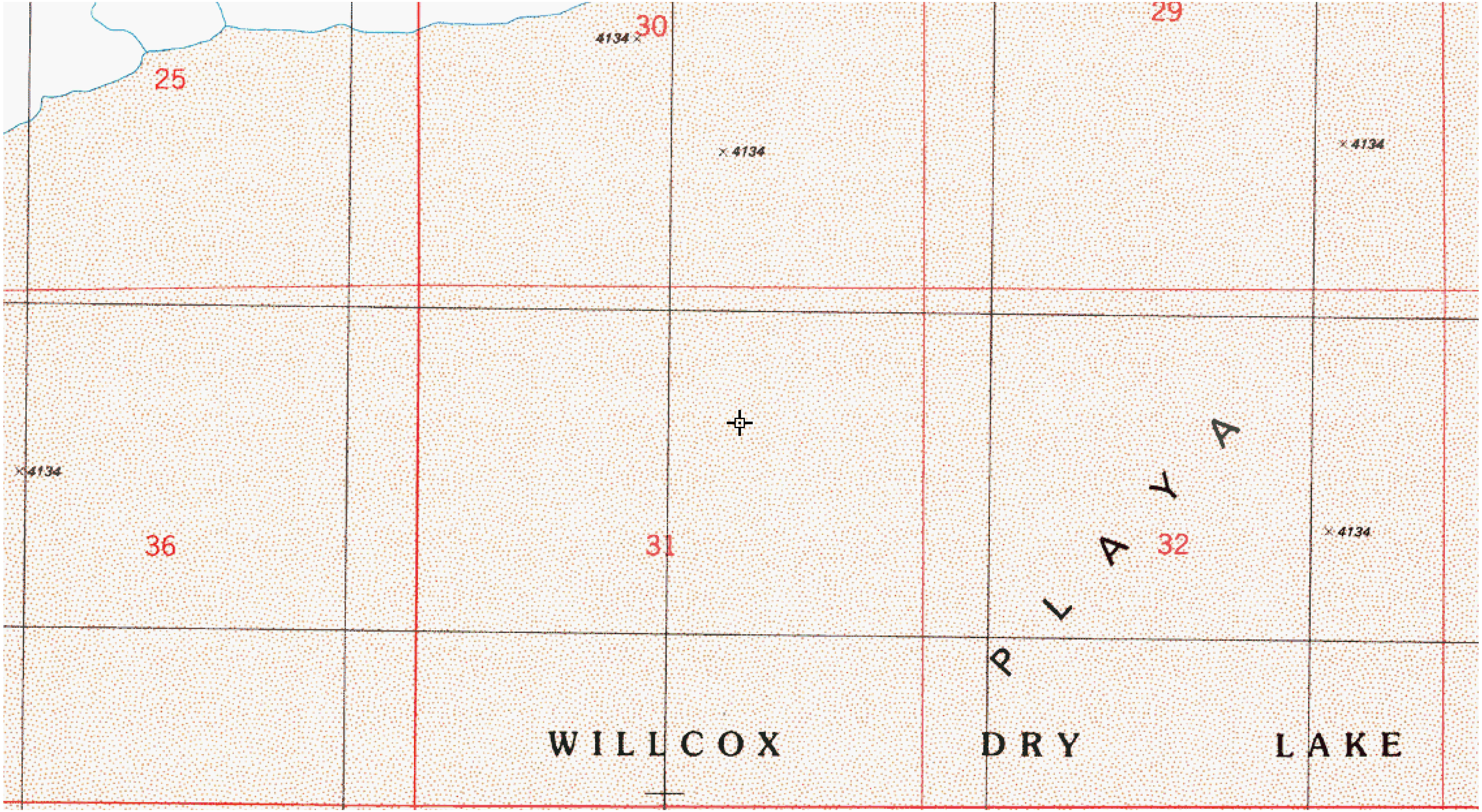
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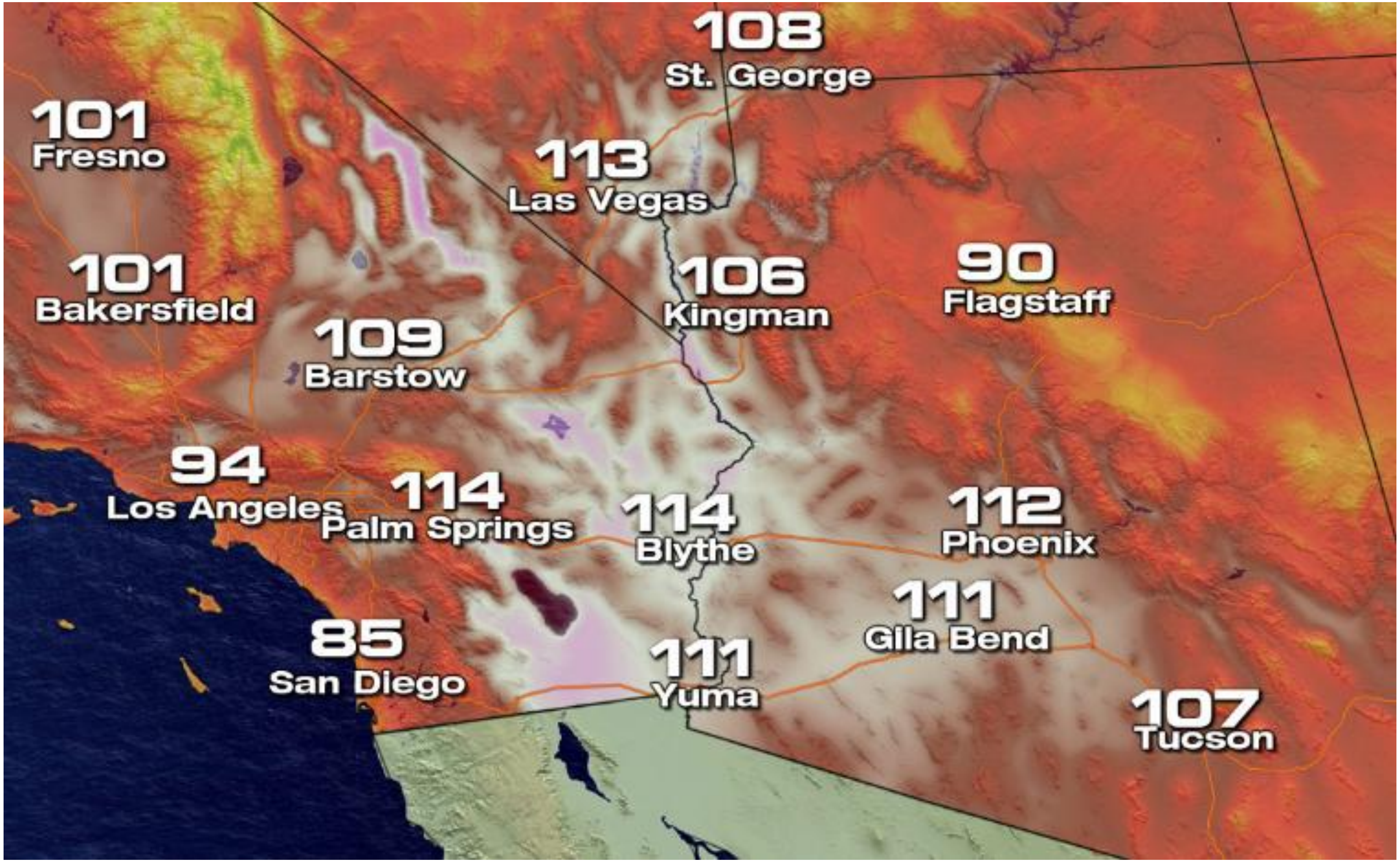
PLAYA
WILLCOX DRY LAKE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Palm Springs
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Blythe
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Phoenix
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Gila Bend
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San Diego
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Yuma
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