Integration in R^3: Riemann Sums and Integration Over Rectangular Regions

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A rectangular region R in the xy-plane can be defined using compound inequalities, where x and y are each bound by constants such that $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$. Let $z = f(x, y)$ be a continuous function defined over a rectangular region R in the xy-plane.

The notation

$$\iint_R f(x,y)\,dA$$

represents the **double integral** of $z = f(x, y)$ over R.

The dA represents "area element", and is either $dy\,dx$ or $dx\,dy$. Thus, we can write

$$\iint_R f(x,y)\,dA = \int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y)\;dy\;dx = \int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y)\;dx\;dy.$$

Note that the bounds a_1 and a_2 correspond with the differential dx, and bounds b_1 and b_2 correspond with dy.

The value of a double integral can be approximated by **Riemann sums** adapted to the two-dimensional case.

Interval $a_1 \leq x \leq a_2$ is subdivided into m subdivisions (not necessarily of equal size) and interval $b_1 \leq y \leq b_2$ is subdivided into n subdivisions (again, not necessarily of equal size).

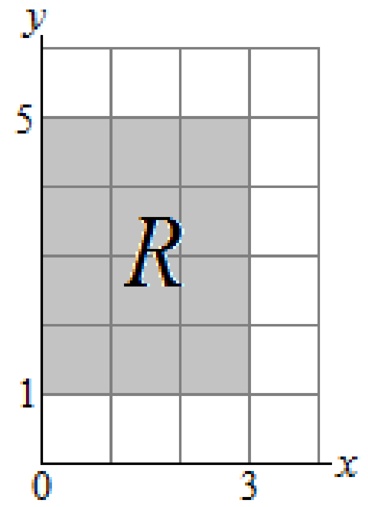
If we define indices $1 \leq i \leq m$ and $1 \leq j \leq n$, then we have a way to identify a particular subdivision within region *R*.

For example, if $a_1 \leq x \leq a_2$ is subdivided into 4 subdivisions and $b_1 \leq y \leq b_2$ is subdivided into 5 subdivisions, then (x_2, y_3) is a representative point within the 2nd subdivision of the *x*-interval and the 3rd subdivision of the *y*-interval, and $f(x_2, y_3)$ is the function evaluated at (x_2, y_3).

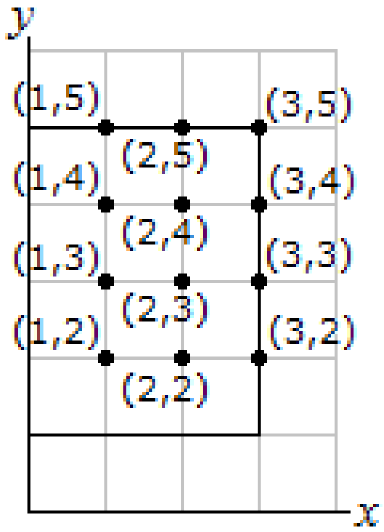
Using this scheme, a double integral can be approximated by a double sum over i and j:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_i, y_j\right)\Delta y\,\Delta x \ \text{ or } \sum_{j=1}^{n}\sum_{i=1}^{m} f\left(x_i, y_j\right)\Delta x\,\Delta y\,.$$

Example 1: Use Riemann Sums to approximate $\iint_R x^2y\,dA$ where R is the rectangle $0 \le x \le 3$ and $1 \le y \le 5$ in the *xy* plane. Subdivide the region *R* into subregions each with length 1 to a side, and from each subregion, choose *x* and *y* to be the "upper right" corner.



Solution: The rectangular region *R* is shown at right, subdivided into subregions, so that $\Delta A = \Delta x\,\Delta y = (1)(1) = 1$. There are 12 such subregions.



Then choose a representative point (x_i, y_j) within each subregion. In this example, we choose (x_i, y_j) to be the "upper right" point within each subregion (this is an arbitrary choice. We could choose the "lower left" or the "middle point", and so on). Here, $1 \le i \le 3$ and $2 \le j \le 5$, the bounds chosen for convenience.

Next, evaluate the integrand $z = f(x, y) = x^2 y$ at the representative points $\left(x_i, y_j\right)$:

$$\begin{array}{lll} f(1,5) = 5 & f(2,5) = 20 & f(3,5) = 45 \\ f(1,4) = 4 & f(2,4) = 16 & f(3,4) = 36 \\ f(1,3) = 3 & f(2,3) = 12 & f(3,3) = 27 \\ f(1,2) = 2 & f(2,2) = 8 & f(3,2) = 18 \end{array}$$

Visually, we have a surface $z = f(x, y) = x^2 y$ "above" the *xy*-plane. Each subregion in *R* is the base of a rectangular box whose height is the function value shown in the table above. Each box has a volume of $f\left(x_i, y_j\right) dA$. Since $dA = dx\, dy = (1)(1) = 1$ in each case, each box has volume $f\left(x_i, y_j\right) \times 1$, or simply $f\left(x_i, y_j\right)$. The value of $\iint_R x^2 y\, dA$ is approximated by the sum of the volumes of the rectangular boxes contained within it. Thus,

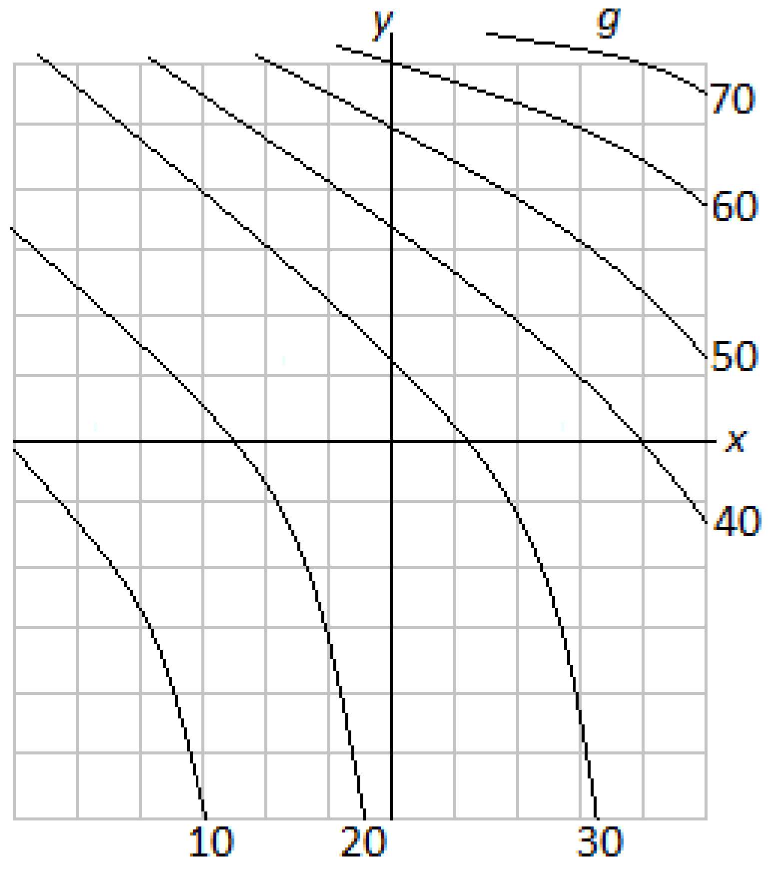
$$\begin{aligned} \iint_R x^2 y\, dA &\approx \sum_{i=1}^{3} \sum_{j=2}^{5} f\left(x_i, y_j\right) \Delta y\, \Delta x \\ &= 2 + 8 + 18 + 3 + 12 + 27 + 4 + 16 + 36 + 5 + 20 + 45 \\ &= 196. \end{aligned}$$

Note that if we chose the representative point to be the lower-left corner of each subregion, we would find that the Riemann Sum is 50.

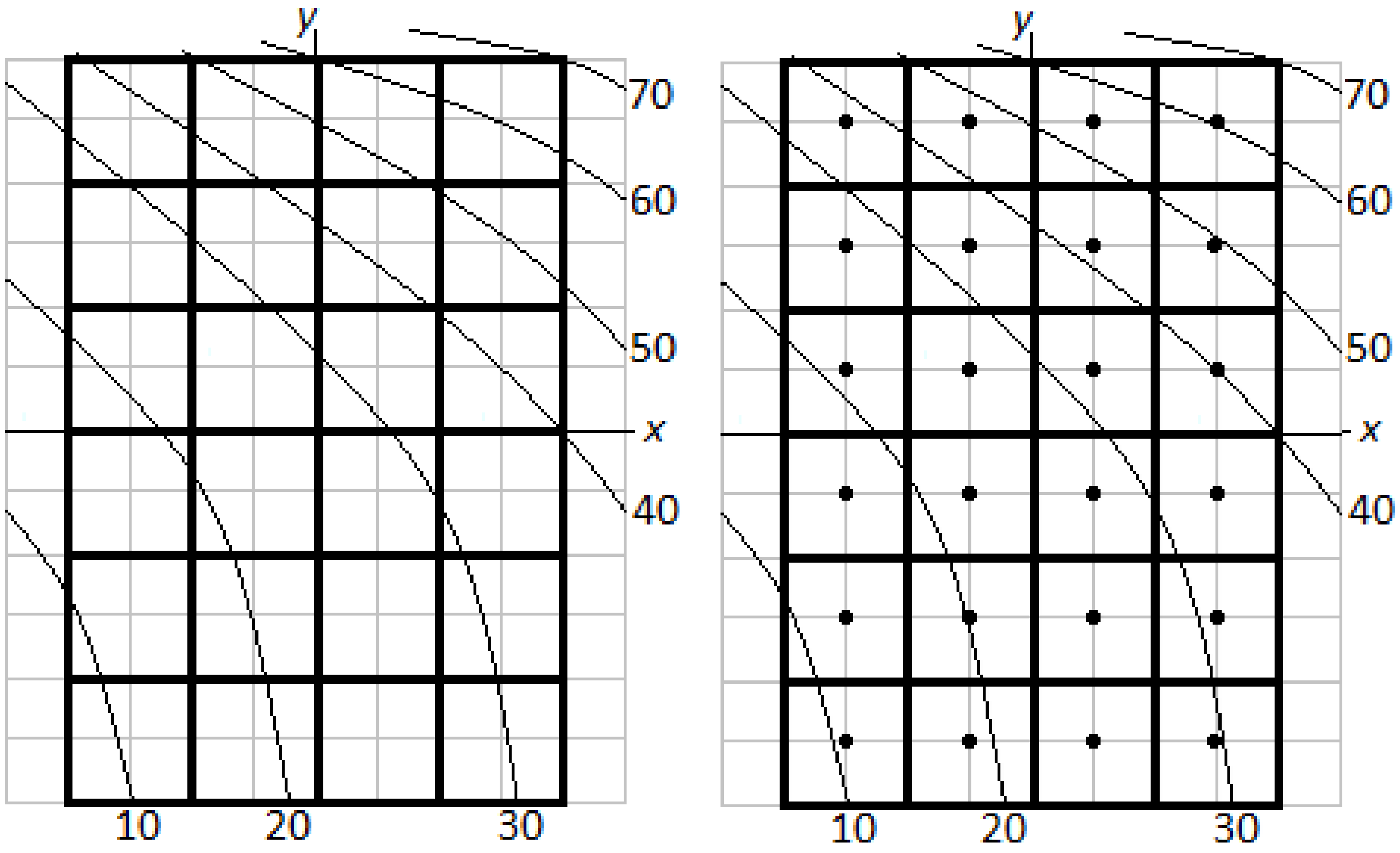
The mean, $\frac{196+50}{2} = 123$, is a reasonable approximation of $\iint_R x^2y\,dA$.

Example 2: Use Riemann Sums to approximate $\iint_R g(x,y)\,dA$, where g is shown by the contour map.

Let the region of integration R be given by $-4 \le x \le 4$, $-6 \le y \le 6$, and let $\Delta x = 2$ and $\Delta y = 2$. Use the middle point within each subregion.



Solution: The region R is identified and then subdivided into 2×2 subregions (lower left, boldfaced). Then the middle point (x_i, y_j) from within each subregion is identified (lower right):



The values of $z = g(x, y)$ are estimated from the contour map. For example, in the top tier of subregions, reading left to right and using the middle points, the values of g are approximately $g(-3,5) = 37$, $g(-1,5) = 46$, $g(1,5) = 55$ and $g(3,5) = 60$.

Each of these subregions is the base of a rectangular box whose heights are given by the $z_i = g(x_i, y_j)$ values. Each box then has a volume of $g(x_i, y_j)\, dA$. Since $dA = (2)(2) = 4$, each box has a volume of $g(x_i, y_j) \times 4$.

The approximate values of $g(x_i, y_j)$ are shown below in an array that matches the orientation of the subregions in the previous figure:

37	46	55	60
27	34	42	49
22	27	33	40
16	23	28	34
13	20	25	31
11	18	25	29

Thus, the approximate value of $\iint_R g(x,y)\,dA$ is the sum of all the $g(x_i, y_j)$ values in the array above, multiplied by 4:

$$\iint_R g(x,y)\,dA \approx 4\begin{pmatrix} 37+46+55+60+27+34+42+49+22+27+33+40 \\ +16+23+28+34+13+20+25+31+11+18+25+29 \end{pmatrix},$$

which is about 2,980 cubic units.

A **double integral** is evaluated "inside out"—that is, the inside integral is evaluated first, then that result becomes the integrand of the outer integral, which is then evaluated.

Example 3: Evaluate $\iint_R x^2y\,dA$ where R is the rectangle $0 \le x \le 3$ and $1 \le y \le 5$.

Solution: We can choose either the $dy\,dx$ ordering or the $dx\,dy$ ordering. Let's choose $dA = dx\,dy$. Thus, we have

$$\iint_R x^2y\,dA = \int_1^5 \int_0^3 x^2y\,dx\,dy\,.$$

Integrate the inner integral with respect to *x*, treating *y* as a constant:

$$\int_0^3 x^2y\,dx = \left[\frac{1}{3}x^3y\right]_0^3 = \frac{1}{3}y[3^3 - 0^3] = 9y.$$

Now we integrate the result with respect to *y*:

$$\int_1^5 9y\,dy = \left[\frac{9}{2}y^2\right]_1^5 = \frac{9}{2}(5^2 - 1^2) = 108.$$

If we chose $dA = dy\,dx$, we have the following:

$$\int_0^3 \int_1^5 x^2 y\, dy\, dx\,.$$

The inner integral is determined first with respect to *y*, treating *x* as a constant temporarily:

$$\int_1^5 x^2 y\, dy = x^2\left[\frac{1}{2}y^2\right]_1^5 = \frac{1}{2}x^2[(5)^2 - (1)^2] = \frac{1}{2}x^2(24) = 12x^2.$$

This result is now integrated with respect to *x*:

$$\int_0^3 12x^2\ dx = [4x^3]_0^3 = 4[(3)^3 - (0)^3] = 4(27) = 108.$$

Both orderings of the differentials gives the same result, 108, as expected. This is the volume of the solid bounded below by the region of integration *R* and above by the surface $z = x^2 y$.

Example 4: The density of a city's population is given by $P(x, y) = 0.2x^2 + 0.1y^3$, where x and y are in miles, and P is on thousands of people per square mile. Assume that the city is a rectangle measuring 6 miles east to west (x), and 4 miles north to south (y), and that $x = 0$ and $y = 0$ is the southwestern corner of the city's boundaries. Find the city's population.

Solution: The city's population is given by the double integral:

$$\int_0^4 \int_0^6 (0.2x^2 + 0.1y^3)\, dx\, dy\,.$$

Evaluating the inside integral with respect to x first, we have

$$\int_0^6 (0.2x^2 + 0.1y^3)\, dx = \left[\frac{0.2}{3}x^3 + 0.1xy^3\right]_0^6$$

$$= \left(\frac{0.2}{3}(6)^3 + 0.1(6)y^3\right) - \left(\frac{0.2}{3}(0)^3 + 0.1(0)y^3\right)$$

$$= 14.4 + 0.6y^3.$$

This is then integrated with respect to *y*:

$$\int_0^4 (14.4 + 0.6y^3)\, dy = \left[14.4y + \frac{0.6}{4}y^4\right]_0^4$$

$$= \left(14.4(4) + \frac{0.6}{4}(4)^4\right) - \left(14.4(0) + \frac{0.6}{4}(0)^4\right)$$

$$= 96.$$

Thus, the city has about 96,000 people within its boundaries.

The **average value** of a multivariable function $z = f(x, y)$ over a region R is given by

$$f_{av} = \frac{1}{A(R)} \iint_R f(x, y)\, dA,$$

where $A(R)$ is the area of region R.

Example 5: Find the average value of the result in the previous example and explain its meaning in context.

Solution: The region R has an area of $(6)(4) = 24$ square miles.

Thus, the average value of $P(x, y) = 0.2x^2 + 0.1y^3$ over R is $P_{av} = \frac{1}{24}(96) = 4$.

The city has an average density of about 4,000 people per square mile.