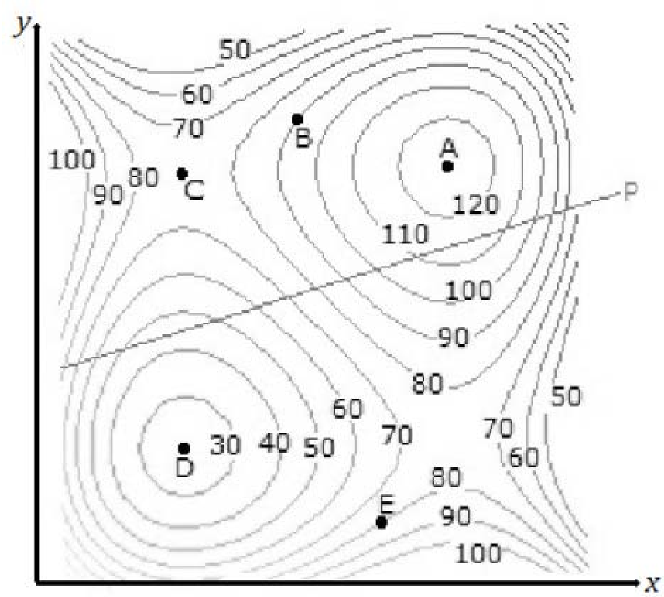
Math 267 Practice Sheet – Midterm (Test 2)

1. What is the domain of $f(x, y) = \frac{1}{\sqrt{2x-y}}$?
2. What is the domain of $g(x, y) = \ln(x^2 + 4y)$?
3. Find all first and second-order partial derivatives of $f(x, y, z) = x^2y^3z^5$.
4. Find the slope of the tangent line to $f(x, y) = x^3y - y^2$ when $x = 1$ and $y = 2$, in the direction of $x = 4$, $y = 3$.
5. Find the direction of the steepest slope of $g(x, y) = \frac{x}{y^2}$ at $x = 3$, $y = -2$. Then find the slope.
6. Find a vector normal to the surface $h(x, y) = 3xy + y^2 - x^3$ at $x = 4$, $y = 2$.
7. Find the equation of the tangent plane in #6.
8. Larry is measuring a circular cylinder. He measures the height to be 8 m, with a tolerance of ±2 cm, and the radius as 3 m, with a tolerance of ±3 cm. Using differentials, find the approximate range of tolerance in the volume of this solid.
9. Let $g(x, y) = x^3 + y^3 - 3x - 12y + 1$. Find all critical points and classify them as min, max or saddle points.
10. Consider the map below. Assume C is a saddle, D is a minimum and A is a maximum, and that the surface is everywhere differentiable.



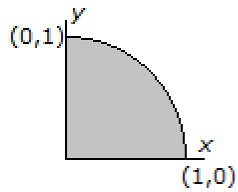
a) Write in the sign (+, –, 0) for the following partial derivatives.

$$f_x(A) = \_\_\_\_ \quad f_x(B) = \_\_\_\_ \quad f_x(E) = \_\_\_\_$$
$$f_y(A) = \_\_\_\_ \quad f_y(B) = \_\_\_\_ \quad f_y(E) = \_\_\_\_$$

b) Suppose P is a path (constraint). Approximate the minimum and maximum values of f constrained to path P.

c) On the map, draw in the gradient vector at E.

11. Let $f(x,y) = 3\sin(x^2 - 4y)$. Find ∇f.
12. Suppose $y = f\big(x(s,t), y(s,t)\big)$, and suppose that $\frac{\partial f}{\partial t} = 10, \frac{\partial f}{\partial x} = 2, \frac{\partial f}{\partial y} = 1, \frac{\partial x}{\partial s} = -3, \frac{\partial x}{\partial t} = 4$ and $\frac{\partial y}{\partial s} = 5$. Find the value of $\frac{\partial y}{\partial t}$.
13. Find the volume below $z = 2e^{x^2+y^2}$ over the region in the *xy*-plane shown below.



14. Let $z = f(x,y) = 2x^2 + xy - \frac{1}{4}y^4 + 4x - 2y + 1$. Find the equation of the tangent plane at $x_0 = -2$ and $y_0 = 3$, then use it to estimate $f(-2.05, 3.1)$. Do not use a calculator.
15. Find $\int_{-1}^{2}\int_{3}^{4} x(1-y)\ dy\, dx$.
16. Rewrite as a single double-integral: $\int_0^2\int_0^{2x} xy\,dy\ dx + \int_2^6\int_0^{6-x} xy\,dy\ dx$.
17. Using only geometry, find the volume below $f(x,y) = \sqrt{1 - x^2 - y^2}$ over the region in #13.
18. Find the volume contained below the paraboloid $z = 4 - x^2 - y^2$ and above the *xy*-plane.
19. Reverse the order of integration: $\int_0^9\int_{-\sqrt{9-y}}^{\sqrt{9-y}} dx\ dy$.
20. Evaluate #19.

Answers. (report errors to surgent@asu.edu)

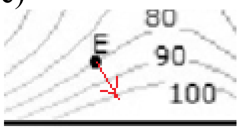
1. $\{(x,y)|y < 2x\}$
2. $\left\{(x,y)\middle|y > -\frac{1}{4}x^2\right\}$
3. $f_x = 2xy^3z^5, f_y = 3x^2y^2z^5, f_z = 5x^2y^3z^4,$
$$\begin{array}{lll} f_{xx} = 2y^3z^5 & f_{xy} = 6xy^2z^5 & f_{xz} = 10xy^3z^4 \\ f_{yx} = 6xy^2z^5 & f_{yy} = 6x^2yz^5 & f_{yz} = 15x^2y^2z^4 \\ f_{zx} = 10xy^3z^4 & f_{zy} = 15x^2y^2z^4 & f_{zz} = 20x^2y^3z^3 \end{array}$$
4. Direction is $v = \langle 3,1\rangle$ so we need the unit direction vector, $\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\rangle$. The gradient of f is $\nabla f = \langle f_x, f_y\rangle = \langle 3x^2y, x^3 - 2y\rangle$. Evaluated at (1,2), we have $\nabla f(1,2) = \langle 6,-3\rangle$. Thus, the slope will be $\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\rangle \cdot \langle 6,-3\rangle = \frac{18}{\sqrt{10}} - \frac{3}{\sqrt{10}} = \frac{15}{\sqrt{10}} \approx 4.743$.
5. The direction of the steepest slope is the gradient of g evaluated at the given point. Thus, $\nabla g = \langle \frac{1}{y^2}, -\frac{2x}{y^3}\rangle \to \nabla g(3,-2) = \langle \frac{1}{4}, \frac{3}{4}\rangle$. The slope is $\left|\langle \frac{1}{4}, \frac{3}{4}\rangle\right| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2} = \frac{\sqrt{10}}{4} \approx 0.791$.
6. Gradient vectors are always orthogonal to a contour line, so we create a 4-dimensional function $H(x,y,z) = 3xy + y^2 - x^3 - z$. Note that $z = 3xy + y^2 - x^3$ is a contour of H, so that $\nabla H = \langle 3y - 3x^2, 3x + 2y, -1\rangle$, and at (4,2), we have $\nabla H(4,2) = \langle -42,16,-1\rangle$. Also acceptable is $\langle 42,-16,1\rangle$.
7. The point is $(4,2,-36)$ so we have $-42(x-4) + 16(y-2) - (z+36) = 0$.
8. The volume is $V(r,h) = \pi r^2 h$, so therefore, $dV = 2\pi rh\,dr + \pi r^2\,dh$. Evaluating, we have $dV = 2\pi(3)(8)(0.03) + \pi(3)^2(0.02) = \pm 5.089$ cubic meters.

9. (1,2,-17) min, (-1,2,-13) saddle, (1,-2,15) saddle, (-1,-2,19) max.
10. a)

$$\begin{array}{lll} f_x(A) = 0 & f_x(B) = + & f_x(E) = + \\ f_y(A) = 0 & f_y(B) = - & f_y(E) = - \end{array}$$

b) Min, about 45, max about 115

c)



Orthogonal to the contour, positive slope.

11. $\nabla f = \langle 6x\cos(x^2 - 4y), -12\cos(x^2 - 4y) \rangle$.
12. $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \to 10 = (2)(4) + (1)\frac{\partial y}{\partial t} \to \frac{\partial y}{\partial t} = 10 - 8 = 2$. Here, y and f are interchangeable.
13. Use polar bounds. The bounds are $0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}$.

$$\int_0^{\pi/2} \int_0^1 2e^{r^2}\, r\, dr\ d\theta = \int_0^{\pi/2} \left[e^{r^2}\right]_0^1 d\theta = \int_0^{\pi/2} (e-1)\, d\theta = (e-1)\int_0^{\pi/2} d\theta = \frac{(e-1)\pi}{2}.$$

14. $f_x(x,y) = 4x + y + 4 \to f_x(-2,3) = -1$, and $f_y(x,y) = x - y^3 - 2 \to f_y(-2,3) = -31$. We also have $z_0 = -31.25$. Thus, the equation of the tangent plane is $z + 31.25 = -(x+2) - 31(y-3)$. Thus,

$$\begin{aligned} z + 31.25 &= -(-2.05 + 2) - 31(3.1 - 3) \\ z + 31.25 &= -(-0.05) - 31(0.1) \\ z + 31.25 &= 0.05 - 3.1 \\ z + 31.25 &= -3.05 \\ z &= -3.05 - 31.25 \\ z &= -34.3 \end{aligned}$$

15. Do the inside integral first: $\int_3^4 x(1-y)\ dy = \left(xy - \frac{1}{2}xy^2\right)_3^4 = (4x - 8x) - \left(3x - \frac{9}{2}x\right) = -\frac{5}{2}x$.

Now integrate with respect to x: $\int_{-1}^2 \left(-\frac{5}{2}x\right) dx = \left(-\frac{5}{4}x^2\right)_{-1}^2 = \left(-\frac{5}{4}(4)\right) - \left(-\frac{5}{4}(1)\right) = -\frac{15}{4}$.
16. $\int_0^4 \int_{y/2}^{6-y} xy\, dx\ dy$
17. The solid is a hemisphere of radius 1, but it's being integrated over a quarter circle, so we have just 1/8 of a sphere. The volume of a sphere of radius 1 is $\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$, then divided by 8, we have $\frac{1}{6}\pi$.
18. The region of integration is a circle of radius 2, so it's advised to use polar. Thus, r ranges between 0 and 2, θ between 0 and 2π. The integrand is $(4 - r^2)r$, where the extra r is from the Jacobian. Thus, we have $\int_0^{2\pi} \int_0^2 (4r - r^3) dr\ d\theta$. The inner integral works out to be 4, the out to be 2π, so the volume is 8π.
19. The region is the parabola $y = 9 - x^2$ above the x-axis, so we have $\int_{-3}^3 \int_0^{9-x^2} dy\ dx$.
20. 36.