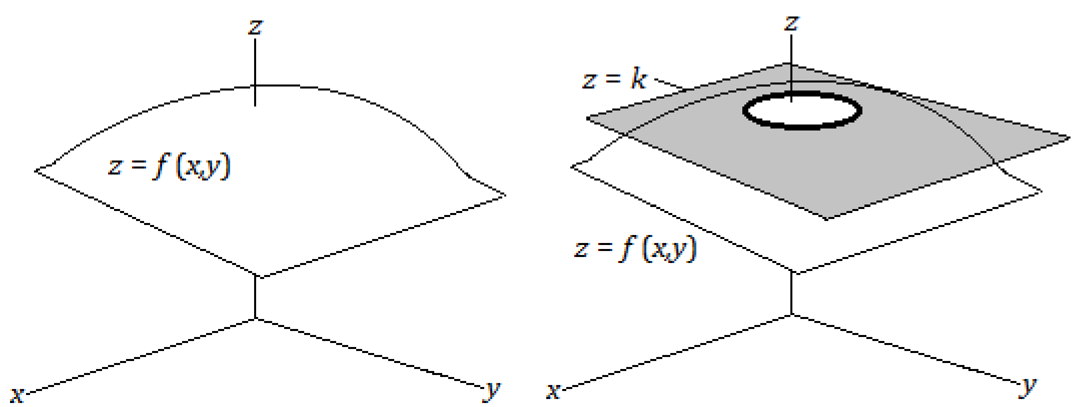
23. Level Curves and Contour Maps

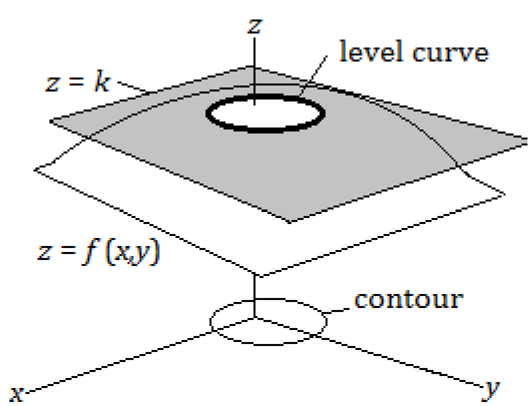
Let $z = f(x, y)$ be a function whose graph is a surface in R^3. Suppose this graph is intersected by a plane $z = k$, parallel to the *xy*-plane. This is equivalent to holding *z* constant and reducing the equation into an implicit function of *x* and *y* only—*i.e.* written $f(x, y) = k$.

The intersection of $z = f(x, y)$ with $z = k$ forms a **level curve** on the surface of f. It is a curve where *x* and *y* can vary, but *z* does not change. Imagine standing on a hill, and taking a step such that you neither go uphill nor downhill. If you do this repeatedly, you will (theoretically) walk along a path that is level, and end at the same point from which you started.

A level curve projected onto the *xy*-plane is called a **contour**.



A surface representing $z = f(x, y)$ is shown at left, then intersected by a plane $z = k$.
The bold path is "level" in that the *z*-values on it do not change.

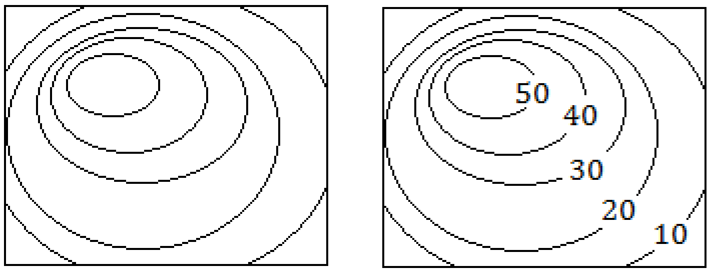


A contour is the projection of a level curve onto the domain plane.

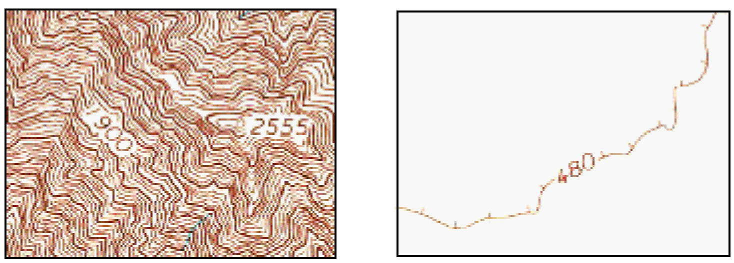
For any surface, sketch a number of level curves (for different values of *z*), whose contours then form a contour map of the surface. With practice, one can locate minimum points, maximum points, saddle points, ridges, valleys, and other "terrain" forms. These maps are usually rendered using software, and they are an effective way to represent a surface in two dimensions.

The usual rules for drawing and reading a contour map include:

- The contours are marked by the *z*-values for reference. For complicated surfaces, the *z*-values may differ by some set value, *e.g.* by multiples of 10. Too many contours, and the map is cluttered, while too few contours, important details may be lost.

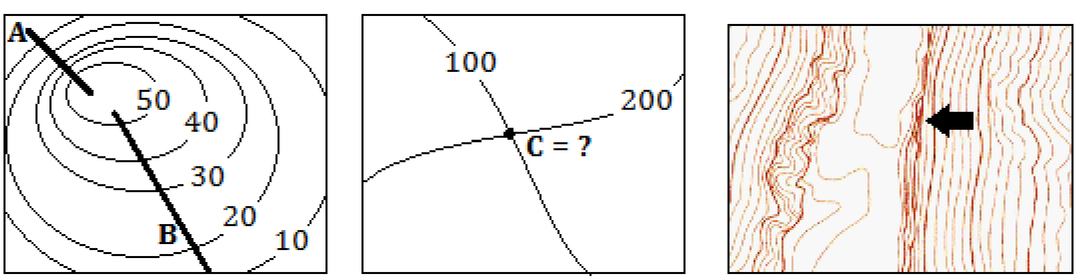


Left: This is a vague contour map. There are no z-values. Is this a hill or a basin?
Right: With *z*-values now in place, we see that this represents a hill.



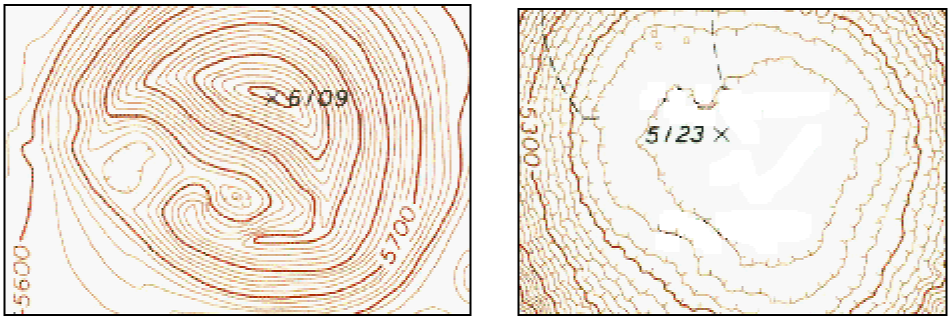
Left: The map is too cluttered. **Right:** Now it's too vague.

- Contour lines for different *z*-values never touch. This would violate the vertical-line rule for functions in R^3. Contours that are close together indicate a large change in *z*, or "steepness", while contours spaced apart indicate flatter terrain. Some topographical maps may show what appears to be contours that "touch". These represent cliffs.



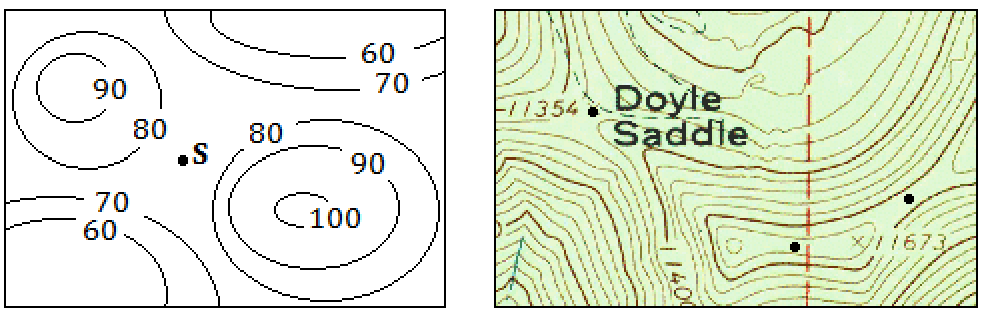
Left: Paths A and B go to the top of the hill, but A is steeper than B.
Center: Contours cannot intersect. What would C's *z*-value be?
Right: These contours are so close they seem to touch.
This is a cliff beside a bluff in the Grand Canyon of Arizona.

- Contours can form closed loops and nest (and or may be nested within) other contours. Hills are found where the nested contours increase in values of *z*. Basins are found where the nested contours decrease in values of *z*.



Left: A hill. Note the darker contours increase by 100, the lighter contours by 20.
Right: A basin. The contour's *z*-values decrease toward the center.
(This is the contour map of Meteor Crater near Winslow, Arizona)
Note that there is a small basin in the left image too, at the middle-left.

- Saddle points are locations where the point may be a minimum along some paths crossing through the point, and a maximum for other paths passing through the point. On most contour maps, a saddle is usually inferred where a contour "pinches" in, then widens again.



Left: A simplified contour map showing two hills (upper left and lower right). A person walking from one hill to the other would pass through S, and this would be the person's lowest point on his walk. Another person walking from the lower left to the upper right would pass through S, and this would be the highest point for this other person's walk. Thus, S is a saddle point. **Right:** Saddles are noted by dots. Note that the contours pinch in near these points.

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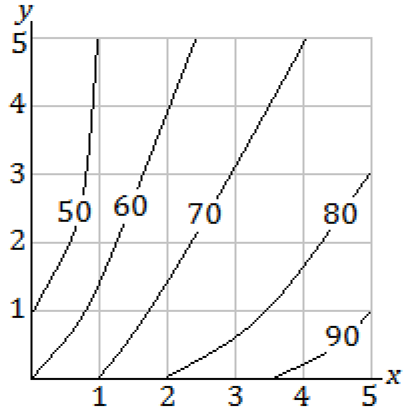
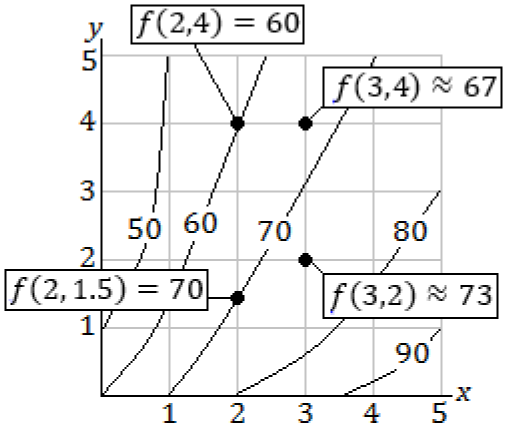
Example 23.1: Given the contour map below for $z = f(x, y)$ where $0 \le x \le 5$ and $0 \le y \le 5$. Find the following: (a) $f(2,4)$; (b) $f(3,2)$; (c) $f(3,4)$; (d) $f(2, y) = 70$.



Solution:

a) Locate $x = 2$ and $y = 4$ and follow the grid lines to where they meet. At this location, it appears that $z = 60$, so we conclude that $f(2,4) = 60$.
b) Using a technique similar to (a), we see that the gridlines for $x = 3$ and $y = 2$ meet between the contours for $z = 70$ and $z = 80$. Here, we surmise that the value of $f(3,2)$ is probably closer to 70 than to 80 since it is situated closer to the contour for $z = 70$. However, to determine z exactly is impossible without a finer contour map. We can say that $f(3,2) \approx 73$, for example. However, any z value such that $70 < z < 80$ would be a plausible (acceptable) answer.
c) Similar to (b), we estimate that $f(3,4) \approx 67$, although any z value such that $60 < z < 70$ is a plausible answer.
d) Following the $z = 70$ contour over the grid for $x = 2$, we see that the corresponding y value is about 1.5. Thus, $f(2, 1.5) = 70$.



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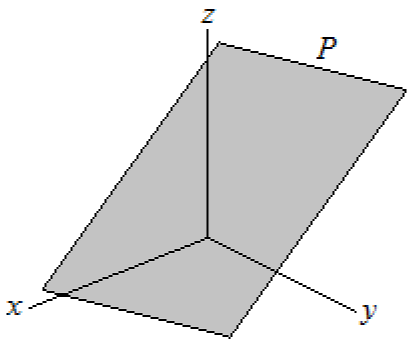
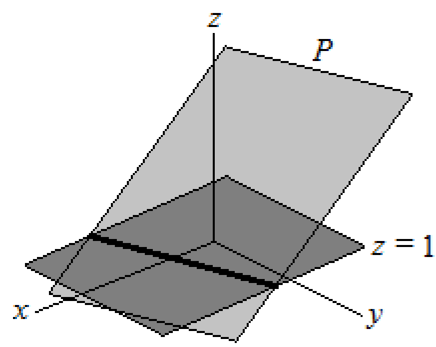
Typical Contour Maps for Planes and Paraboloids

Planes

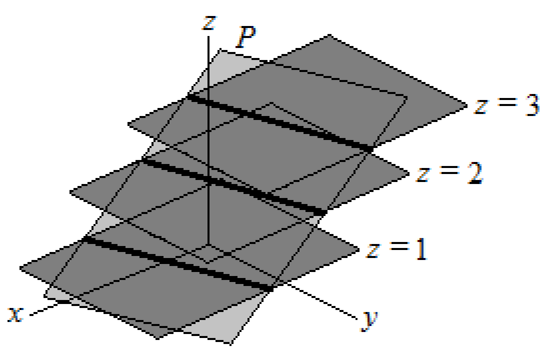
A plane of the form $z = ax + by + c$ has a contour map of equally-spaced lines.



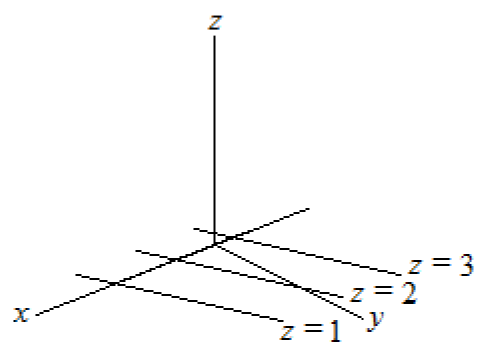
Start with a plane P.



Intersect it with planes of the form $z = k$, parallel to the xy-plane. The dark line is a level curve.



Intersect the plane P a few more times by planes $z = k$, for different values of k, forming more level curves.

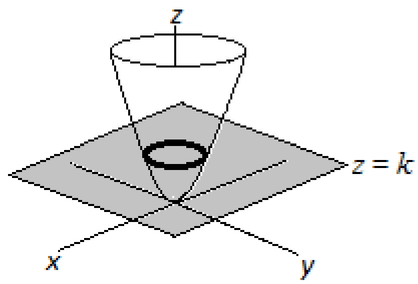


Project these level curves down to the xy-plane. These are now contours. For a plane, its contours will be parallel lines.

Paraboloids

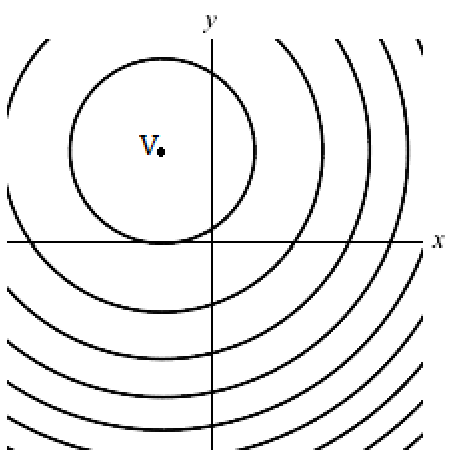
A paraboloid of the form $z = ax^2 + by^2 + cx + dy + e$, where a and b are the same sign, has concentric contours emanating outward from the vertex. If $a = b$, then the contours are circles. Otherwise, the contours are ellipses. If the function contains an *xy* term (while still a paraboloid), then the contours may be angled so that their axes are not parallel to the *x* and *y* axes.

In the simplest case, $z = x^2 + y^2$, an intersection of this surface with a plane $z = k$ forms a level curve that is a circle. Thus, the contour map of this paraboloid will be concentric circles centered at the origin.



The paraboloid $z = x^2 + y^2$ intersected by this surface with a plane $z = k$ forms a circle.

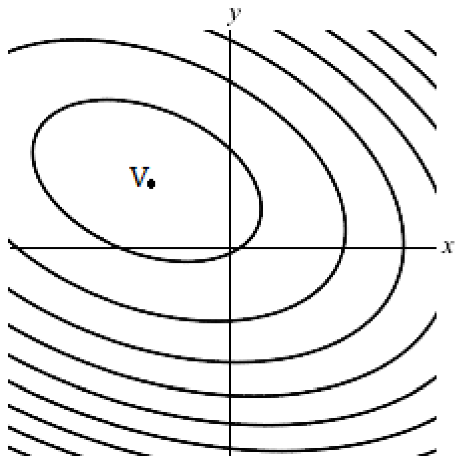
The contour map below represents the paraboloid $z = x^2 + y^2 + 2x - 4y$, whose vertex is $(-1,2)$. Since the coefficients of the quadratic terms are the same, the contours are circles. Note that the farther away from the origin, the contours are spaced closer. This is a trick of perspective. In reality, the contours are spaced evenly when looking along the *z*-axis. However, as *x* and *y* are chosen farther awar from the vertex, the change in *z* is greater. The close-in contours indicate a steeper rate of change.



V = vertex

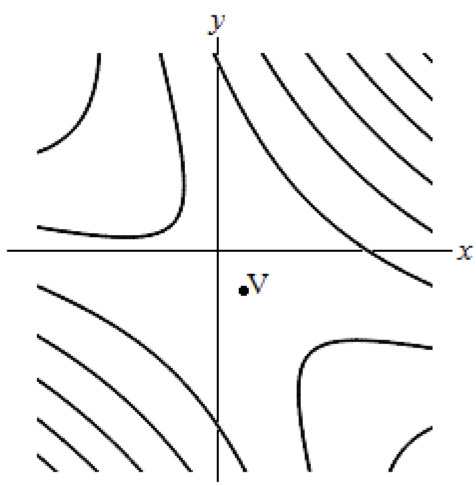
$z = x^2 + y^2 + 2x - 4y$

The contour map below is for the paraboloid $z = x^2 + 2y^2 + xy + 2x - 4y$, whose vertex is approximately $(-1.71, 1.43)$. Note that the coefficients of the quadratic terms are not equal, so that the contours are ellipses. Also, the *xy* term causes the orientation of the major and minor elliptic axes to rotate slightly.



$z = x^2 + 2y^2 + xy + 2x - 4y$

For the surface $z = x^2 + y^2 + 4xy + x$, the vertex is $\left(\frac{1}{6}, -\frac{1}{3}\right)$, but the *xy* term has "deformed" the paraboloid into a saddle-shaped surface. Note the "pinching" effect of the contour lines.



$z = x^2 + y^2 + 4xy + x$

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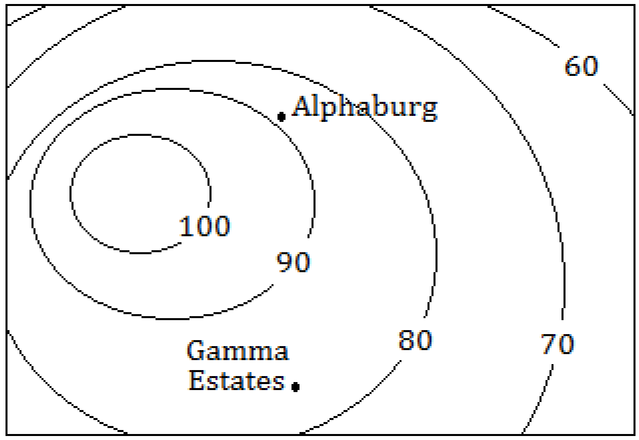
Example 23.2: The image below is a contour map indicating the temperature for certain locations within a rectangular county.



Based on this image, answer the following questions.

a) What is the approximate temperature in Alphaburg?
b) What is the approximate temperature in Gamma Estates?
c) What is the approximate hottest temperature within the county?

Solution:

a) Since the location of Alphaburg lies in the area between the contours representing $z = 80$ degrees and $z = 90$ degrees, its temperature is $80 < z < 90$. However, since the location is close to the $z = 90$ contour, it's reasonable to assume that Alphaburg's temperature is in the high 80s, perhaps 88 or 89 degrees. It is *not* reasonable to assume that Alphaburg's temperature is 90 degrees since the location is not on the 90-degree contour.
b) Gamma Estates' temperature would be within the $80 < z < 90$ range. Since it is closer to the $z = 80$ contour, it is reasonable to assume the temperature is approximately 82 or 83 degrees.
c) The hottest temperature is within the $z = 100$ contour. We can assume that it is in the range $100 < z < 110$. It is not reasonable to assume that the temperature is 110 degrees or higher. An approximate figure of $z = 105$ degrees would be reasonable.

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