Green's Theorem

Let $\mathbf{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ be a vector field in R^2 and assume functions M and N are twice differentiable.

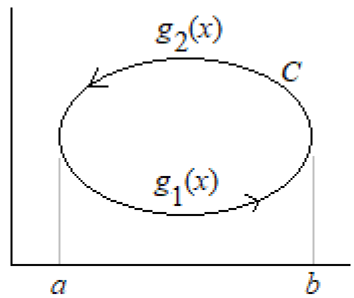
Let C be a closed (starts and ends at the same point) and simple (does not cross itself) loop path in R^2 that is traversed counterclockwise. Let D be the interior of C. (Equivalently, C is the boundary of D).

Green's Theorem:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_D N_x - M_y \, dA.$$

This allows one to find the value of the line integral along path C by instead finding a double integral over the interior D.

Proof:



To the right is a closed simple loop path C traversed counterclockwise.

Treat C as the union of two paths, $y = g_1(x)$ which traverses from a to b, and $y = g_2(x)$, which traverses from b to a.

Note that the parameterization of the path results in the form $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, so that the derivative is $d\mathbf{r} = \mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$, or abbreviated $\langle dx, dy\rangle$.

The line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is equivalent to

$$\int_C \langle M(x,y), N(x,y)\rangle \cdot \langle dx, dy\rangle = \int_C M(x,y)\,dx + N(x,y)\,dy.$$

We'll look at $\int_C M(x,y)\,dx$:

$$\begin{aligned}\int_C M(x,y)\,dx &= \int_a^b M(x,g_1(x))\,dx + \int_b^a M(x,g_2(x))\,dx \\ &= \int_a^b M(x,g_1(x))\,dx - \int_a^b M(x,g_2(x))\,dx \qquad \text{Reversing the order of integration} \\ &= -\left[\int_a^b M(x,g_2(x))\,dx - M(x,g_1(x))\,dx\right]\end{aligned}$$

The integrand in the last integral is a subtraction of two quantities. We can write this as an integral itself, using the trick that $p - q = \int_q^p du$. In the above case, we have

$$M(x,g_2(x)) - M(x,g_1(x)) = \int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\,dy.$$

Note that we develop an integrand in such a way that after antidifferentiation, we're left with the function $M(x,y)$. Also note that the expressions $g_1(x)$ and $g_2(x)$ are evaluated into the y-variable. Thus, we need the derivative of M with respect to y to be the integrand. To do this, we also must include a differential dy. Note that dy and ∂y mean the same thing.

Thus, we have

$$-\left[\int_a^b M\big(x, g_2(x)\big)\, dx - M\big(x, g_1(x)\big)\, dx\right] = -\int_a^b \left(\int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\, dy\right) dx.$$

This last integral is simplified as

$$-\iint_D M_y\, dA.$$

We have shown that

$$\int_C M(x, y)\, dx = -\iint_D M_y\, dA.$$

A similar construction shows that

$$\int_C N(x, y)\, dy = \iint_D N_x\, dA.$$

Combined, we have

$$\begin{aligned}\int_C M(x, y)\, dx + N(x, y)\, dy &= \iint_D N_x\, dA. - \iint_D M_y\, dA \\ &= \iint_D (N_x - M_y)\, dA.\end{aligned}$$