Let $\mathbf{r}(t)$ and $\mathbf{s}(t)$ be two differentiable vector-valued functions in R^3. There is a useful property:

- Derivative of the dot product: $\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{s}(t)] = \frac{d}{dt}\mathbf{r}(t)\cdot\mathbf{s}(t) + \mathbf{r}(t)\cdot\frac{d}{dt}\mathbf{s}(t)$.

Note that it appears similar to the product rule of differentiation.

Theorem: If $|\mathbf{r}(t)|$ is constant for all t, then $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are orthogonal.

Proof: Assume that $|\mathbf{r}(t)| = c$.

Then square both sides: $|\mathbf{r}(t)|^2 = c^2$.

Recall that $|\mathbf{r}(t)|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$: $\mathbf{r}(t)\cdot\mathbf{r}(t) = c^2$.

Now, differentiate both sides:

$$\begin{aligned}
\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] &= \frac{d}{dt}[c^2] \\
\frac{d}{dt}\mathbf{r}(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\frac{d}{dt}\mathbf{r}(t) &= 0 \\
\mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0 \\
2\mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0 \\
\mathbf{r}(t)\cdot\mathbf{r}'(t) &= 0.
\end{aligned}$$

In R^2, the only paths for which $|\mathbf{r}(t)|$ is constant are circular paths. However, in higher dimensions, these paths may not look circular. For example, in R^3, a path where $|\mathbf{r}(t)|$ is constant would look like any path drawn on a sphere. The path itself may be a bunch of squiggles, but it would be embedded on a sphere of constant radius.

Unit Tangent and Unit Normal vectors.

Given $\mathbf{r}(t)$, we define the **unit tangent** vector to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Now, consider its derivative, $\mathbf{T}'(t)$. Since $\mathbf{T}(t)$ is a unit vector, $|\mathbf{T}(t)| = 1$ (a constant), and thus it meets the condition for the theorem. This means that $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal: $\mathbf{T}(t)\cdot\mathbf{T}'(t) = 0$. However, even though $\mathbf{T}(t)$ is a unit vector, its derivative usually is not. Thus, we define the **unit normal** to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

Since $\mathbf{T}(t)$ and $\mathbf{T}'(t)$ are orthogonal, then so are $\mathbf{T}(t)$ and $\mathbf{N}(t)$. Unit normal $\mathbf{N}$ always points into the curve of a path, whereas $\mathbf{T}$ points in the direction of travel.

Arc Length

The length of the path traced out by $\mathbf{r}(t)$ is given by

$$s = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Letter *s* is often used for arc length. If *a* and *b* are constants, then the definite integral above is a constant. However, we can make arc length *s* into a function of *t*. We let *t* be the upper bound and use a dummy variable *u* within the integral:

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du.$$

Differentiating both sides with respect to *t* and using the Fundamental Theorem of Calculus, we have

$$\frac{d}{dt}s(t) = |\mathbf{r}'(t)|, \qquad \text{or equivalently,} \qquad ds = |\mathbf{r}'(t)|\, dt.$$

The *dt* and *ds* Segmentations of a Path

When defining a path using parameter variable *t*, the path usually does not subdivide into subsegments of equal length. For example, $\mathbf{r}(t) = \langle t, t^2 \rangle$ traces out the common parabola. Note that when $t = 0$, the path is a point (0,0), and when $t = 1$, the path is at point (1,1) and when $t = 2$, the path is at point (2,4). The segment from (0,0) to (1,1) is obviously shorter than the path from (1,1) to (2,4). A particle moving along this path will need to speed up as *t* increases. We call this the ***dt* segmentation**.

In many applications, we prefer the tangent vector remain the same length. This removes the variable of speed from the problem (essentially, one less thing to worry about). The easiest way to do this is to use the unit tangent, $\mathbf{T}(t)$. Using the parabola example, this would be

$$\mathbf{T}(t) = \langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \rangle.$$

The important thing to note is that while **T** now has constant length 1, the little segments of the parabola will not necessarily be of length 1. But they will be constant, because a particle moving along the path moves at a constant speed. This is called the ***ds* segmentation**. It is usually not important to know the length of these little segments, but that they're constant.

In later applications, we set up a problem by assuming an equal segmentation of a path, which is easy to declare but difficult to do algebraically. We then use the formula $ds = |\mathbf{r}'(t)|\, dt$ to convert back into the "easier" *dt* segmentation. Thus, the *ds* segmentation is used for the set-up, but the actual solutions will come via the *dt* segmentation, where strategic substitutions have been made along the way.