36. Double Integration over Non-Rectangular Regions of Type II

When establishing the bounds of a double integral, visualize an arrow initially in the positive *x* direction or the positive *y* direction. A region of **Type II** is one in which there may be ambiguity as to where this arrow enters or exits the region. In such cases, more than one double integral will be required. It is possible that a region may be Type II in one ordering of integration, but Type I in another ordering.

Example 36.1: Rewrite the following integral expression

$$\int_0^2 \int_0^{x^2} f(x,y)\,dy\,dx + \int_2^6 \int_0^{6-x} f(x,y)\,dy\,dx$$

in the *dx dy* order of integration.

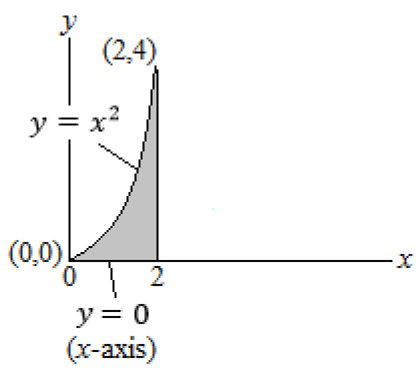
Solution: Let's look at one double integral at a time. We start with the first,

$$\int_0^2 \int_0^{x^2} f(x,y)\,dy\,dx.$$

It may be helpful to write in equations for each bound:

$$\int_{x=0}^{x=2} \int_{y=0}^{y=x^2} f(x,y)\,dy\,dx.$$

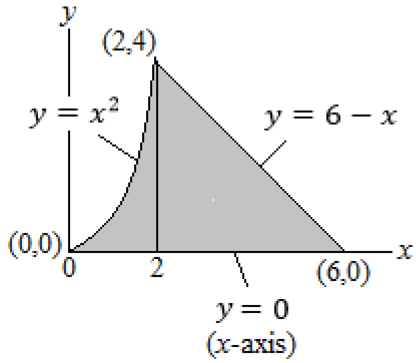
The bounds of the inner integral suggests a region with a lower bound of $y = 0$ (the *x*-axis), and an upper bound of $y = x^2$. Then, the bounds of the outer integral suggest that *x* must be contained in the interval $0 \leq x \leq 2$. We obtain the following region:



Now, we examine the second double integral, where the bounds have been written as equations:

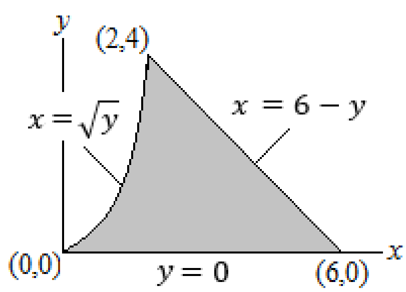
$$\int_{x=2}^{x=6} \int_{y=0}^{y=6-x} f(x,y)\, dy\, dx$$

This suggests a region bounded below by $y = 0$ (the x-axis) and above by the line $y = 6 - x$, then the bounds of the outer integral suggest that $2 \leq x \leq 6$. This region is sketched alongside the one already sketched. All vertex or extreme points are also noted:



In the *dy dx* ordering of integration, this is a Type II region. An arrow drawn in the positive *y*-direction enters the region at the *x*-axis ($y = 0$), but may exit through the parabola $y = x^2$ or the line $y = 6 - x$. This ambiguity is why this region is considered Type II in the *dy dx* ordering, and why two double integrals are necessary to describe the region properly.

If the order of integration is reversed to $dx\, dy$, then there is no ambiguity as to where an arrow drawn in the positive *x*-direction would enter and exit the region. The same region is drawn below, but now the boundaries are stated with variable *x* isolated:



An arrow drawn in the positive *x* direction enters the region at $x = \sqrt{y}$ and exits at $x = 6 - y$. The bounds for *y* are 0 to 4. In this ordering, the region is Type I, and one double integral is sufficient to describe this region:

$$\int_{0}^{4} \int_{\sqrt{y}}^{6-y} f(x,y)\, dx\, dy.$$

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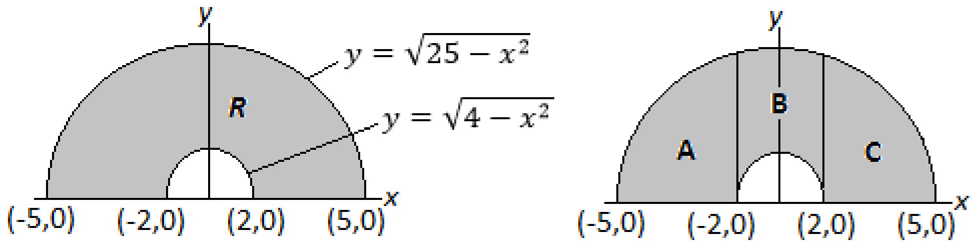
Example 36.2: Set up a double integral over region *R* that is outside a circle of radius 2 centered at the origin, inside a circle of radius 5 centered at the origin, such that *y* is non-negative. Use the $dy\ dx$ ordering and use $f(x,y)$ as the integrand.

Solution: A sketch shows that this region has the following appearance, and is Type II. Vertical lines are placed where *R* is split into smaller Type I regions, labeled *A*, *B* and *C*, reading left to right:



For region *A*, we have

$$\int_{-5}^{-2}\int_{0}^{\sqrt{25-x^2}} f(x,y)\ dy\ \ dx.$$

For region *B*, we have

$$\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{25-x^2}} f(x,y)\ dy\ \ dx.$$

For region *C*, we have

$$\int_{2}^{5}\int_{0}^{\sqrt{25-x^2}} f(x,y)\ dy\ \ dx.$$

Summing, we have

$$\int_{-5}^{-2}\int_{0}^{\sqrt{25-x^2}} f(x,y)\ dy\, dx + \int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{25-x^2}} f(x,y)\ dy\, dx + \int_{2}^{5}\int_{0}^{\sqrt{25-x^2}} f(x,y)\ dy\, dx.$$

Regions that are formed by circles are usually better solved using polar and cylindrical coordinates, which are discussed later.

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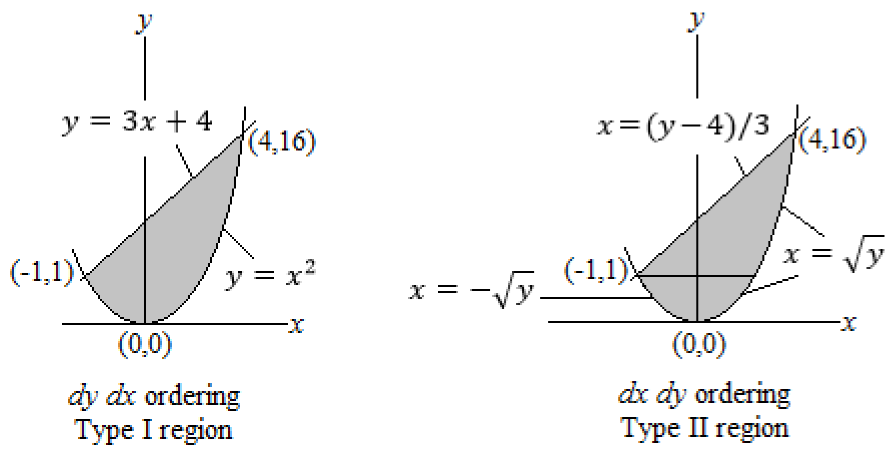
Example 36.3: Reverse the order of integration:

$$\int_{-1}^{4}\int_{x^2}^{3x+4} g(x,y)\,dy\,dx.$$

Solution: The region is shown below. In the $dy\,dx$ ordering of integration, it is Type I (at left).



However, as a $dx\,dy$ ordering of integration, an arrow drawn in the positive x direction may enter the region through the line $x = \frac{y-4}{3}$ (assuming $1 < y \le 16$) or through the curve $x = -\sqrt{y}$ (assuming $0 \le y < 1$). Because of the ambiguity as to where such an arrow could enter the region, this is a Type II region.

We split the region at $y = 1$, forming two smaller Type I regions, as shown above right, with the bounding curves now written with x isolated. Thus, in the $dx\,dy$ ordering of integration, we have

$$\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} g(x,y)\,dx\,dy + \int_{1}^{16}\int_{(y-4)/3}^{\sqrt{y}} g(x,y)\,dx\,dy.$$

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37. Double Integration in Polar Coordinates

Regions that are formed by circles are better described using polar coordinates. If (r, θ) represents a point in the plane, then r is the distance from the point to the origin, and θ represents the angle that a ray from the origin to the point makes with the positive *x*-axis. The usual conversion formulas between rectangular (x, y) coordinates to polar (r, θ) coordinates are:

$$(x, y) \text{ to } (r, \theta): \begin{cases} r^2 = x^2 + y^2 \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases} \qquad (r, \theta) \text{ to } (x, y): \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}.$$

Circular regions (or portions thereof) in the *xy*-plane can be described using polar coordinates where $a \le r \le b$ and $c \le \theta \le d$, and a, b, c and d are constants. Such regions are called **polar rectangles**.

To establish bounds for r, visualize an arrow that starts at the origin and extends outward. The value of r at which the arrow enters the region is the lower bound a, and the value of r at which this arrow exits the region is the upper bound b. If the region includes the origin, then the lower bound for r is 0. Since r is a radius, it is never negative.

To establish bounds for θ, visualize a ray attached at the origin that "sweeps" through the region in a counterclockwise manner. The angle at which θ enters the region is the lower bound c, and the value at which θ exits the region is the upper bound d. The "sweep" must always be done in a counterclockwise manner, so it may be necessary to allow θ to be negative in order to preserve the ordering of the values c and d.

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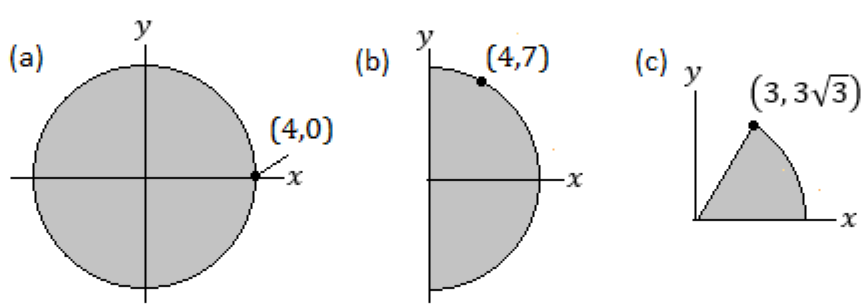
Example 37.1: Describe the following regions using polar coordinates.



Solution:

(a) The point (4,0) in rectangular coordinates suggests that this circle has a radius of 4. Since the region includes the origin, we have a lower bound of $r = 0$, and since the circle has radius 4, we have an upper bound of $r = 4$. Thus, the interval for r is $0 \le r \le 4$. Since this is an entire circle, we have the interval for θ as $0 \le \theta \le 2\pi$.

(b) The point (4,7) in rectangular coordinates allows us to determine the radius, which is $r = \sqrt{4^2 + 7^2} = \sqrt{65}$. The region includes the origin. Thus, $0 \leq r \leq \sqrt{65}$. Meanwhile, this semicircle is swept by a ray that would start at the negative *y*-axis and sweep counterclockwise to the positive *y*-axis. Thus, the interval for θ is $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

(c) The point $(3, 3\sqrt{3})$ in rectangular coordinates allows us to find the radius, which is $r = \sqrt{3^2 + (3\sqrt{3})^2} = \sqrt{9 + 27} = \sqrt{36} = 6$. Since the region also contains the origin, we have $0 \leq r \leq 6$. The region is swept by a ray that starts at the positive *x*-axis, so $\theta = 0$. To find the upper bound for θ, observe that the point $(3, 3\sqrt{3})$ lies on this ray. Since $x = 3$ and $y = 3\sqrt{3}$, we have $\theta = \arctan\left(\frac{3\sqrt{3}}{3}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$ radians. Thus, $0 \leq \theta \leq \frac{\pi}{3}$.

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Integration in Polar Coordinates: The standard form for integration in polar coordinates is

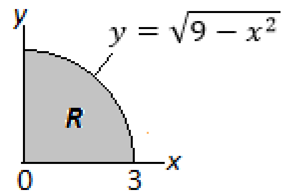
$$\int_c^d \int_a^b f(r\cos\theta, r\sin\theta)\ r\,dr\,d\theta,$$

where $a \leq r \leq b$ and $c \leq \theta \leq d$. The area element is $r\,dr\,d\theta$, the r being the **Jacobian** of integration.

Example 37.2: Evaluate

$$\int_0^3 \int_0^{\sqrt{9-x^2}} xy\,dy\,dx.$$

Solution: A sketch of the region of integration shows it to be a quarter circle in the first quadrant, centered at the origin, with radius 3.



This integral is better solved using polar coordinates. The bounds of integration are $0 \leq r \leq 3$ and $0 \leq \theta \leq \frac{\pi}{2}$. Furthermore, we substitute $x = r\cos\theta$ and $y = r\sin\theta$, and exchange $dy\,dx$ with $r\,dr\,d\theta$:

$$\int_0^3 \int_0^{\sqrt{9-x^2}} xy\,dy\,dx = \int_0^{\pi/2} \int_0^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \int_0^{\pi/2} \int_0^3 r^3 \cos\theta\sin\theta\ dr\,d\theta.$$

The inside integral is evaluated first:

$$\int_0^3 r^3 \cos\theta \sin\theta \; dr = \cos\theta \sin\theta \left[\frac{1}{4}r^4\right]_0^3$$
$$= \frac{81}{4}\cos\theta \sin\theta.$$

This is then integrated with respect to θ, using *u*-*du* substitution, with $u = \sin\theta$ and $du = \cos\theta$:

$$\frac{81}{4}\int_0^{\pi/2} \cos\theta \sin\theta \; d\theta = \left[\frac{81}{8}\sin^2\theta\right]_0^{\pi/2}$$
$$= \frac{81}{8}[1^2 - 0]$$
$$= \frac{81}{8}.$$

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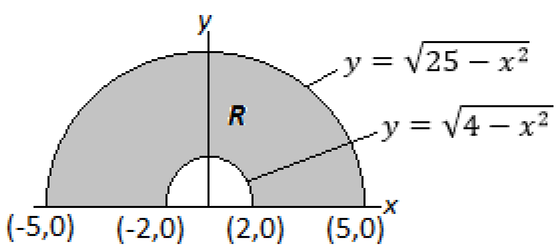
Example 37.3: Evaluate

$$\int_{-5}^{-2}\int_0^{\sqrt{25-x^2}} x^2\,dy\,dx + \int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{25-x^2}} x^2\,dy\,dx + \int_2^5\int_0^{\sqrt{25-x^2}} x^2\,dy\,dx.$$

Solution: The region of integration as suggested by the bounds in the three integrals is shown below (See Example 36.2):



This region is better described using polar coordinates, where $2 \le r \le 5$ and $0 \le \theta \le \pi$. Replace the integrand x^2 with $(r\cos\theta)^2 = r^2\cos^2\theta$, and the area element $dy\,dx$ with $r\,dr\,d\theta$. In doing so, the three double integrals above, in rectangular coordinates, are equivalent to one double integral, in polar coordinates, with constant bounds:

$$\int_0^{\pi}\int_2^5 r^2\cos^2\theta \; r\,dr\,d\theta, \qquad \text{which simplifies to} \qquad \int_0^{\pi}\int_2^5 r^3\cos^2\theta \; dr\,d\theta.$$

The inside integral is evaluated first:

$$\int_2^5 r^3 \cos^2\theta \; dr = \cos^2\theta \left[\frac{1}{4}r^4\right]_2^5$$
$$= \frac{1}{4}\cos^2\theta \,[(5)^4 - (2)^4]$$
$$= \frac{609}{4}\cos^2\theta.$$

This expression is then integrated with respect to θ. To antidifferentiate $\cos^2\theta$, use the identity $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$:

$$\int_0^\pi \frac{609}{4}\cos^2\theta \, d\theta = \frac{609}{4}\int_0^\pi \left(\frac{1}{2}(1+\cos 2\theta)\right) d\theta$$
$$= \frac{609}{8}\int_0^\pi 1+\cos 2\theta \;\; d\theta$$
$$= \frac{609}{8}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^\pi$$
$$= \frac{609}{8}\left[\left(\pi + \frac{1}{2}\sin 2(\pi)\right) - \left(0 + \frac{1}{2}\sin 2(0)\right)\right]$$
$$= \frac{609}{8}\pi\,.$$

In this example, it is clearly faster and preferable to integrate using polar coordinates rather than perform three double integrals using rectangular coordinates (and probable trigonometric substitutions along the way).

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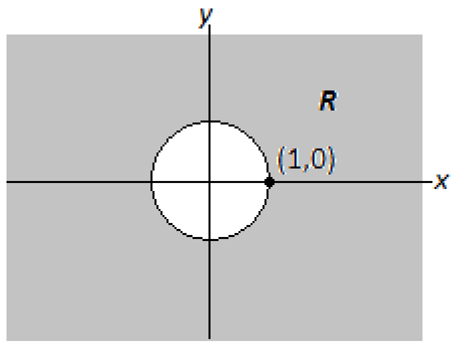
Example 37.4: Evaluate

$$\iint_R \frac{1}{(1+x^2+y^2)^2}\, dA,$$

where R is the region in the xy-plane outside the circle of radius 1 centered at the origin.

Solution: Below is a sketch of the region of integration R.



An arrow drawn from the origin outward enters region R when $r = 1$. Since the region extends forever, use ∞ as the upper bound for r. Thus, $1 \le r < \infty$. The bounds for θ are $0 \le \theta \le 2\pi$.

The double integral, using polar coordinates, is

$$\int_0^{2\pi}\int_1^{\infty} \frac{1}{(1+r^2)^2}\, r\, dr\ d\theta.$$

The inside integral is evaluated using *u*-*du* substitution, where $u = 1 + r^2$ and $du = 2r\, dr$. We have

$$\begin{aligned}
\int_1^{\infty} \frac{1}{(1+r^2)^2}\, r\, dr &= \lim_{d\to\infty}\int_1^{d} \frac{r}{(1+r^2)^2}\, dr \\
&= \lim_{d\to\infty}\left[\frac{1}{2}\int_1^{d} \frac{1}{(1+r^2)^2}\, 2r\, dr\right] \\
&= \frac{1}{2}\lim_{d\to\infty}\left[-\frac{1}{1+r^2}\right]_1^d \\
&= \frac{1}{2}\lim_{d\to\infty}\left[\left(-\frac{1}{1+(d)^2}\right)-\left(-\frac{1}{1+(1)^2}\right)\right] \\
&= \frac{1}{2}\left[0-\left(-\frac{1}{2}\right)\right] = \frac{1}{4}.
\end{aligned}$$

The integral with respect to θ is now evaluated:

$$\frac{1}{4}\int_0^{2\pi} d\theta = \frac{1}{4}[\theta]_0^{2\pi} = \frac{2\pi}{4} = \frac{\pi}{2}.$$

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Example 37.5: Find the volume of the ellipsoid $x^2 + \frac{1}{9}y^2 + z^2 = 1$.

Solution: We must decide which variable should be isolated. Note that solving for *y* yields

$$y = \pm 3\sqrt{1 - x^2 - z^2}\,.$$

Furthermore, when $y = 0$, then the ellipsoid forms a circle, $x^2 + z^2 = 1$, on the *xz*-plane. We will integrate $y = f(x, z)$ over a region of integration R that is the disk $x^2 + z^2 \leq 1$. In rectangular coordinates, we have the double integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(3\sqrt{1-x^2-z^2} - \left(-3\sqrt{1-x^2-z^2}\right)\right)\,dz\;dx.$$

This looks challenging, so instead, we use polar coordinates in place of variables *x* and *z*. Region *R* is now defined by $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$, and the integrand is now written $y = \pm 3\sqrt{1 - r^2}$. The double integral in polar coordinates is now

$$\int_0^{2\pi}\int_0^{1}\left(3\sqrt{1-r^2} - \left(-3\sqrt{1-r^2}\right)\right)\,r\,dr\,d\theta.$$

This simplifies to

$$6\int_0^{2\pi}\int_0^{1}\sqrt{1-r^2}\;r\,dr\,d\theta.$$

Note: It is also possible to use symmetry to set up the integral, noting that the volume between the *xz*-plane and $y = 3\sqrt{1 - r^2}$ is the same as the volume between the *xz*-plane and $y = -3\sqrt{1 - r^2}$. Thus, we could use $3\sqrt{1 - r^2}$ as the integrand, and double the result. Either way, the $\sqrt{1 - r^2}$ remains in the integrand, the 3 moves to the front and is doubled, so that we arrive at the same integral as above.

The inside integral is evaluated using *u*-*du* substitution, with $u = 1 - r^2$ and $du = -2r\,dr$:

$$\int_0^1 \sqrt{1 - r^2}\; r\,dr = \left[-\frac{1}{3}(1 - r^2)^{3/2}\right]_0^1$$
$$= 0 - \left(-\frac{1}{3}\right)$$
$$= \frac{1}{3}.$$

This is then integrated with respect to θ:

$$6\int_0^{2\pi} \left(\frac{1}{3}\right) d\theta = 2\int_0^{2\pi} d\theta = 4\pi.$$

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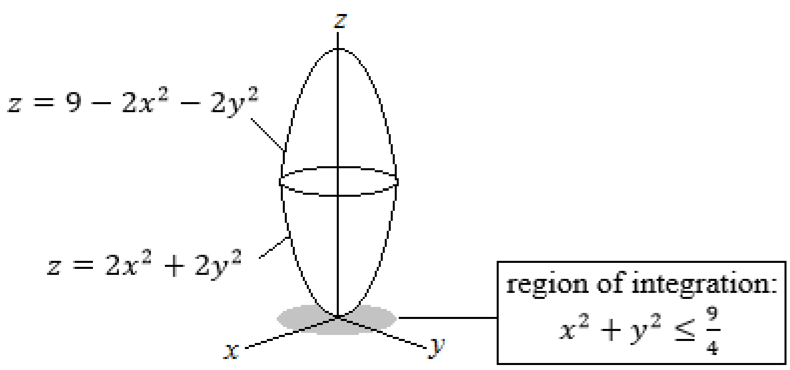
Example 37.6: Find the volume of the solid bounded by $z = 2x^2 + 2y^2$ and $z = 9 - 2x^2 - 2y^2$.

Solution: Setting the two functions equal, we have

$$2x^2 + 2y^2 = 9 - 2x^2 - 2y^2$$
$$4x^2 + 4y^2 = 9$$
$$x^2 + y^2 = \frac{9}{4}.$$

Thus, the region of integration is the disk $x^2 + y^2 \le \frac{9}{4}$, which can be described in polar coordinates as $0 \le r \le \frac{3}{2}$ and $0 \le \theta \le 2\pi$. Below is a sketch of the solid along with the region of integration:



The integrand is the "top" boundary ($z = 9 - 2x^2 - 2y^2$) subtracted by the "bottom" boundary ($z = 2x^2 + 2y^2$). This is

$$\begin{aligned} 9 - 2x^2 - 2y^2 - (2x^2 + 2y^2) &= 9 - 4x^2 - 4y^2 \\ &= 9 - 4(x^2 + y^2) \\ &= 9 - 4r^2. \end{aligned}$$

The volume is found by evaluating

$$\int_0^{2\pi} \int_0^{3/2} (9 - 4r^2)\, r\, dr\, d\theta\,, \qquad \text{or simplified as:} \qquad \int_0^{2\pi} \int_0^{3/2} (9r - 4r^3)\, dr\, d\theta.$$

For the inner integral, we have

$$\begin{aligned} \int_0^{3/2} (9r - 4r^3)\, dr &= \left[\frac{9}{2} r^2 - r^4\right]_0^{3/2} \\ &= \frac{9}{2}\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^4 - 0 \\ &= \frac{243}{16}. \end{aligned}$$

Finally, the volume is

$$\begin{aligned} \int_0^{2\pi} \frac{243}{16}\, d\theta &= \frac{243}{16} \int_0^{2\pi} d\theta \\ &= \frac{243}{16}(2\pi) \\ &= \frac{243\pi}{8} \text{ units}^3. \end{aligned}$$

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Example 37.7: A sphere of radius 10 is intersected by a circular cylinder of radius 6 such that the cylinder and the sphere share a common axis of symmetry (that is, the cylinder's axis of symmetry intersects the sphere through one of the sphere's diameters). Find the volume of the outer ring-shaped solid, which consists of the material inside the sphere but outside of the cylinder.

Solution: Centering everything at the origin and using the *z*-axis as the line of symmetry, we can define the sphere as $x^2 + y^2 + z^2 = 100$, or $z = \sqrt{100 - x^2 - y^2}$. The sphere intersects the *xy*-plane and creates a disk $x^2 + y^2 \leq 100$, but the cylinder then removes the inner portion, everything inside a circle of radius 6. Thus, using polar coordinates, the bounds are $6 \leq r \leq 10$ and $0 \leq \theta \leq 2\pi$. The integrand is $z = \sqrt{100 - x^2 - y^2} = \sqrt{100 - r^2}$. The volume of this solid is given by

$$2\int_0^{2\pi}\int_6^{10}\sqrt{100 - r^2}\ r\,dr\,d\theta.$$

The leading 2 represents the fact that the integral, as shown, will determine the volume between the sphere's surface and the *xy*-plane. We need to double the result to find the entire solid's volume.

The inside integral is evaluated:

$$\begin{aligned}\int_6^{10}\sqrt{100 - r^2}\ r\,dr &= \left[-\frac{1}{3}(100 - r^2)^{3/2}\right]_6^{10}\\ &= \left(-\frac{1}{3}(100 - (10)^2)^{3/2}\right) - \left(-\frac{1}{3}(100 - (6)^2)^{3/2}\right)\\ &= \left(-\frac{1}{3}(100 - 100)^{3/2}\right) - \left(-\frac{1}{3}(100 - 36)^{3/2}\right)\\ &= 0 - \left(-\frac{1}{3}(64)^{3/2}\right)\\ &= \frac{512}{3}.\end{aligned}$$

Then, the outer integral is evaluated. We have

$$\begin{aligned}2\int_0^{2\pi}\frac{512}{3}\,d\theta &= \frac{1024}{3}(2\pi)\\ &= \frac{2048\pi}{3}\ \text{units}^3.\end{aligned}$$

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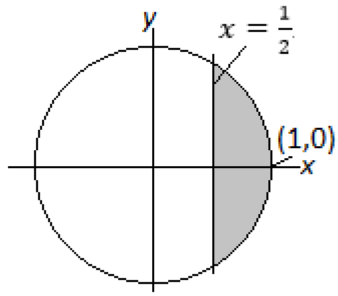
Example 37.8: Find the area inside a circle or radius 1 centered at the origin, to the right of the vertical line $x = \frac{1}{2}$.

Solution: The region is shown below:



The circle's equation is $x^2 + y^2 = 1$, or $y = \pm\sqrt{1 - x^2}$. Using a single integral, the shaded area is given by

$$2\int_{1/2}^{1} \sqrt{1 - x^2}\ dx.$$

However, this integral requires a trigonometric substitution. Instead, we try a different approach, using a double integral in polar coordinates. The boundaries are redefined: the circle is $r = 1$, and for the line $x = \frac{1}{2}$, and using the substitution $x = r\cos\theta$, we have $r\cos\theta = \frac{1}{2}$, or $r = \frac{1}{2\cos\theta}$. Setting the two equations equal, we solve to determine the bounds for θ:

$$1 = \frac{1}{2\cos\theta} \quad \text{so that} \quad \cos\theta = \frac{1}{2}. \quad \text{Thus,} \quad \theta = \pm\arccos\left(\frac{1}{2}\right).$$

Since the region is symmetric with the positive *x*-axis, we use 0 as a lower bound for θ and add a leading factor of 2 to the integral to double the result. Thus, the bounds are $0 \le \theta \le \arccos\left(\frac{1}{2}\right)$.

Visualizing an arrow drawn outward from the origin, it enters the region at the line $r = \frac{1}{2\cos\theta}$, and exits at the circle $r = 1$, so the bounds for r are $\frac{1}{2\cos\theta} \le r \le 1$. Since this is an area integral, we use a 1 for the integrand. However, recall that in polar coordinates, the Jacobian r will also be present in the integrand.

$$2\int_{0}^{\arccos(1/2)} \int_{1/(2\cos\theta)}^{1} 1\, r\, dr\, d\theta.$$

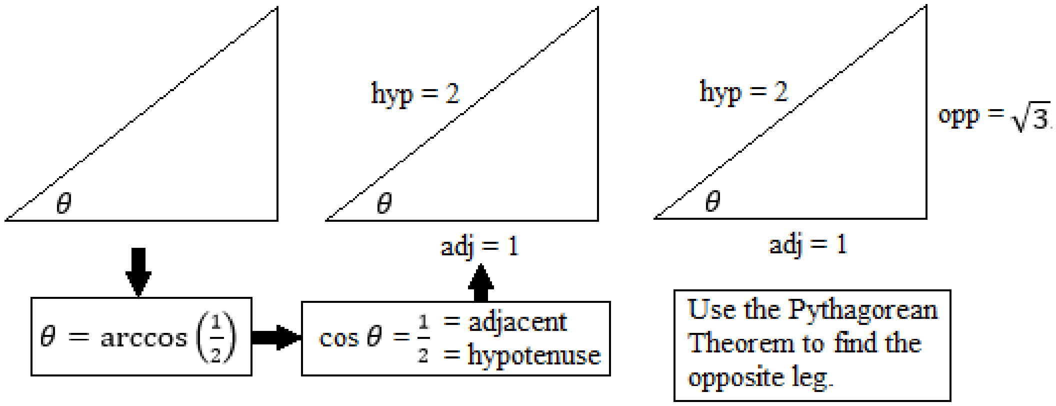
The inside integral is evaluated:

$$\int_{1/(2\cos\theta)}^{1} r\,dr = \left[\frac{1}{2}r^2\right]_{1/(2\cos\theta)}^{1}$$
$$= \frac{1}{2}\left(1^2 - \left(\frac{1}{2\cos\theta}\right)^2\right)$$
$$= \frac{1}{2}\left(1 - \frac{1}{4}\sec^2\theta\right).$$

Recall that $\int \sec^2 u\ du = \tan u + C$. Thus, we now integrate this expression with respect to θ:

$$2\int_0^{\arccos(1/2)} \frac{1}{2}\left(1 - \frac{1}{4}\sec^2\theta\right)\,d\theta = \int_0^{\arccos(1/2)} \left(1 - \frac{1}{4}\sec^2\theta\right)\,d\theta$$
$$= \left[\theta - \frac{1}{4}\tan\theta\right]_0^{\arccos(1/2)}$$
$$= \left(\arccos\left(\frac{1}{2}\right) - \frac{1}{4}\tan\left(\arccos\left(\frac{1}{2}\right)\right)\right) - 0.$$

The expression $\tan\left(\arccos\left(\frac{1}{2}\right)\right)$ can be simplified. If $\theta = \arccos\left(\frac{1}{2}\right)$, then this suggests a right triangle with hypotenuse of length 2, adjacent leg of length 1, and by Pythagoras' Theorem, an opposite leg of length $\sqrt{3}$.



Thus, we have $\tan\theta = \frac{\text{opposite leg}}{\text{adjacent leg}} = \frac{\sqrt{3}}{1} = \sqrt{3}$, so therefore, $\tan\left(\arccos\left(\frac{1}{2}\right)\right) = \sqrt{3}$. The area of the region is $\arccos\left(\frac{1}{2}\right) - \frac{\sqrt{3}}{4}$, or about 0.614 units^2.

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