How the Dot Product is derived from the Law of Cosines

Given two vectors, $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, the dot product is defined two ways:

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \qquad \text{and} \qquad \mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

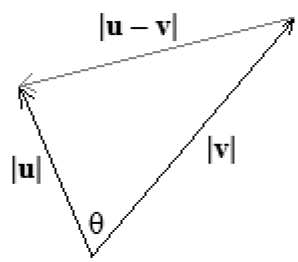
Both derive from the Law of Cosines.

The **Law of Cosines** relates the three sides of any triangle:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

This assumes angle θ is opposite side c.

Let **u** and **v** be two vectors and θ the angle formed when their feet are placed together. The difference vector $\mathbf{u} - \mathbf{v}$ "closes" off the triangle:



Then the Law of Cosines now reads

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta.$$

Using the relationship of the dot product with magnitude, $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$, we have

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2|\mathbf{u}||\mathbf{v}|\cos\theta.$$

Expanding the left side gives

$$\mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} - 2|\mathbf{u}||\mathbf{v}|\cos\theta.$$

Cancelling gives

$$-2\mathbf{u} \cdot \mathbf{v} = -2|\mathbf{u}||\mathbf{v}|\cos\theta.$$

Dividing by -2 gives the first form of the dot product

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta.$$

The second formula relies on the magnitude formula of a vector:

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} \qquad \text{so that} \quad |\mathbf{u}|^2 = u_1^2 + u_2^2 + u_3^2.$$

The difference vector $\mathbf{u} - \mathbf{v}$ is the difference of the corresponding components:

$$\mathbf{u} - \mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

Thus, its magnitude-squared is

$$|\mathbf{u} - \mathbf{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$

Expanded:

$$|\mathbf{u} - \mathbf{v}|^2 = u_1^2 - 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2 + u_3^2 - 2u_3v_3 + v_3^2$$

Regrouping, we have

$$|\mathbf{u} - \mathbf{v}|^2 = \underbrace{(u_1^2 + u_2^2 + u_3^2)}_{|\mathbf{u}|^2} - 2(u_1v_1 + u_2v_2 + u_3v_3) + \underbrace{(v_1^2 + v_2^2 + v_3^2)}_{|\mathbf{v}|^2}$$

Going back to the formula

$$|\mathbf{u} - \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta.$$

We get

$$\underbrace{(u_1^2 + u_2^2 + u_3^2)}_{|\mathbf{u}|^2} - 2(u_1v_1 + u_2v_2 + u_3v_3) + \underbrace{(v_1^2 + v_2^2 + v_3^2)}_{|\mathbf{v}|^2} = |\mathbf{u}|^2 + |\mathbf{v}|^2 - 2|\mathbf{u}||\mathbf{v}|\cos\theta$$

Cancelling leaves

$$-2(u_1v_1 + u_2v_2 + u_3v_3) = -2|\mathbf{u}||\mathbf{v}|\cos\theta.$$

Thus,

$$u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta = \mathbf{u} \cdot \mathbf{v}.$$