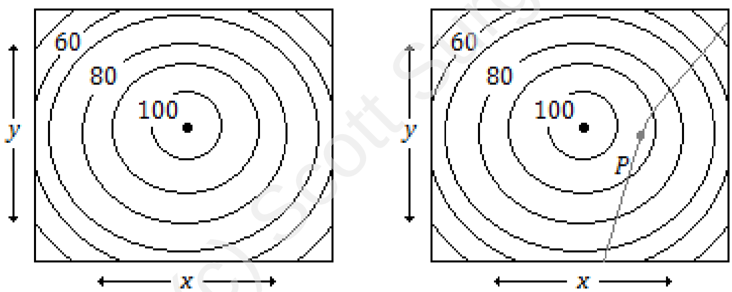
30. Constrained Optimization

The graph of $z = f(x, y)$ is represented by a surface in R^3. Normally, x and y are chosen independently of one another so that one may "roam" over the entire surface of f (within any domain restrictions on x and y). Determining minimum or maximum points on f under this circumstance is called unconstrained optimization.

If x and y are related to one another by an equation, then only one of the variables can be independent. In such a case, we may determine a minimum or maximum point on the surface of f subject to the constraint placed on x and y. This is called **constrained optimization**. Such constraints are usually written where x and y are combined implicitly, $g(x, y) = c$.

Suppose you are hiking on a hill. If there is no restriction on where you may walk, then you are "unconstrained" and you may seek the hill's maximum point. However, if you are constrained to a hiking path, then it is possible to determine a maximum point on the hill, but only that part along the hiking path.



(Left) Unconstrained optimization: The maximum point of this hill is marked by a black dot, and is roughly $z = 105$.
(Right) Constrained optimization: The highest point on the hill, subject to the constraint of staying on path P, is marked by a gray dot, and is roughly $z = 93$.

The two common ways of solving constrained optimization problems is through substitution, or a process called The Method of Lagrange Multipliers (which is discussed in a later section). Using substitution, the biggest challenge is the amount of algebra that may occur.

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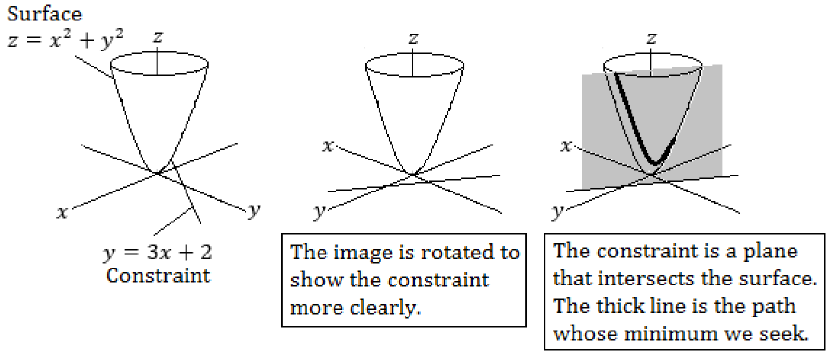
Example 30.1: Find the minimum or maximum point on the surface of $z = f(x, y) = x^2 + y^2$ subject to the constraint $-3x + y = 2$.

Solution: The surface of f is a paraboloid with its vertex (0,0,0) at the origin, opening in the positive z direction (or "up"). Its unconstrained minimum point is (0,0,0). There is no maximum point on this surface.

Now, note that with the constraint $y = 3x + 2$ in place, x and y are no longer independent variables. Once a value for x is chosen, then y is determined. We are now restricted to this "path" on the surface of f.



To find the minimum or maximum point on this paraboloid subject to the constraint $y = 3x + 2$, substitute the constraint into the function f and simplify:

$$f(x, 3x + 2) = x^2 + (3x + 2)^2 = 10x^2 + 12x + 4.$$

Differentiating, we have $f'(x) = 20x + 12$, and to find the critical value of x, we set $f'(x) = 0$:

$$20x + 12 = 0 \qquad \text{which gives} \qquad x = -\frac{3}{5}.$$

Observe that $f(x) = 10x^2 + 12x + 4$ is a parabola in R^2 that opens upward. Thus, the critical value for x will correspond to the minimum point on this parabola. Find y by substitution into the constraint:

$$y = 3\left(-\frac{3}{5}\right) + 2 = \frac{1}{5}.$$

Lastly, we find z:

$$z = f\left(-\frac{3}{5}, \frac{1}{5}\right) = \left(-\frac{3}{5}\right)^2 + \left(\frac{1}{5}\right)^2 = \frac{10}{25} = \frac{2}{5}.$$

Thus, the minimum *point* on the surface of f subject to the constraint $y = 3x + 2$ is $\left(-\frac{3}{5}, \frac{1}{5}, \frac{2}{5}\right)$. The minimum *value* of z on the surface of f subject to the constraint $y = 3x + 2$ is $= \frac{2}{5}$.

Example 30.2: Find the minimum and maximum points on the surface of $z = f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 4$.

Solution: Solving for one of the variables in the constraint, we have $y = \pm\sqrt{4 - x^2}$, which implies that $-2 \le x \le 2$. This is substituted into the function $f(x, y) = xy$:

$$f\left(x, \sqrt{4 - x^2}\right) = x\sqrt{4 - x^2} \quad \text{or} \quad f\left(x, -\sqrt{4 - x^2}\right) = -x\sqrt{4 - x^2}.$$

Differentiate. We will start with $f(x) = x\sqrt{4 - x^2}$, using the product rule and combining into a single expression, noting any restrictions:

$$f'(x) = x\left(\frac{-2x}{2\sqrt{4 - x^2}}\right) + \sqrt{4 - x^2} = \frac{4 - 2x^2}{\sqrt{4 - x^2}}, \qquad (x \ne \pm 2).$$

Setting this equal to 0, we can find critical values for *x*. Note that this reduces to showing where the numerator, $4 - 2x^2$, is 0:

$$4 - 2x^2 = 0 \quad \text{gives} \quad x = \pm\sqrt{2}.$$

Substituting this back into the constraint, each *x* value results in two *y* values:

$$\begin{aligned} x = \sqrt{2}&: \ \left(\sqrt{2}\right)^2 + y^2 = 4 \quad \text{gives} \quad y = \pm\sqrt{2}, \\ x = -\sqrt{2}&: \ \left(-\sqrt{2}\right)^2 + y^2 = 4 \quad \text{gives} \quad y = \pm\sqrt{2}. \end{aligned}$$

In this example, repeating the above steps with $f(x) = -x\sqrt{4 - x^2}$ results in the same critical values for *x* and *y* (you verify). Thus, we have four critical points, where the *z*-value is found by evaluating at the given *x* and *y* values:

$$\left(\sqrt{2}, \sqrt{2}, 2\right), \qquad \left(\sqrt{2}, -\sqrt{2}, -2\right), \qquad \left(-\sqrt{2}, \sqrt{2}, -2\right), \qquad \left(-\sqrt{2}, -\sqrt{2}, 2\right).$$

By inspection, the minimum points on $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 4$ are $\left(\sqrt{2}, -\sqrt{2}, -2\right)$ and $\left(-\sqrt{2}, \sqrt{2}, -2\right)$, and the maximum points are $\left(\sqrt{2}, \sqrt{2}, 2\right)$ and $\left(-\sqrt{2}, -\sqrt{2}, 2\right)$.

This is plausible: the surface f is positive in quadrants 1 and 3, negative in quadrants 2 and 4, and symmetric across the origin. The constraint is also symmetric across the origin. Because the constraint is a closed loop, it must achieve both minimum and maximum values.

The values for which the derivative is not defined, $x = \pm 2$, both imply that $y = 0$, and that $f(2,0) = 0$ and $f(-2,0) = 0$. These are clearly neither minimum nor maximum points, and thus can be ignored.

Example 30.3: Find the extreme points of $f(x, y) = x^3 + y^2 + 2xy$ subject to the constraint $x = y^2 - 1$.

Solution: Since x is already isolated in the constraint, substitute it into the function:

$$f(y^2 - 1, y) = (y^2 - 1)^3 + y^2 + 2(y^2 - 1)y.$$

Simplifying, we have a function in one variable:

$$f(y) = y^6 - 3y^4 + 2y^3 + 4y^2 - 2y - 1.$$

Differentiating, we have

$$f'(y) = 6y^5 - 12y^3 + 6y^2 + 8y - 2.$$

We set this to 0 to determine critical values for *y*. The challenge is that we have to somehow glean solutions from this 5th-degree polynomial. In this case, we graph $f'(y)$ and note where it crosses the input axis (if using a graphing calculator, the *y* variable will be renamed *x* in the calculator. Just keep track of this). The critical *y* values are

$$y \approx -1.377, \quad y \approx -0.831 \quad \text{and} \quad y \approx 0.228.$$

Recall that x and y are related by the equation $x = y^2 - 1$. Thus, the corresponding *x* values are:

$$\begin{aligned} y = -1.377&: \quad x = (-1.377)^2 - 1 = 0.896, \\ y = -0.831&: \quad x = (-0.831)^2 - 1 = -0.309, \\ y = 0.228&: \quad x = (0.228)^2 - 1 = -0.948. \end{aligned}$$

(From this point forward, it's accepted that all *x*, *y* and *z* values will be approximated values.)

The full coordinates for the critical points are

$$(0.896, -1.377, 0.148), \quad (-0.309, -0.831, 1.175)$$
$$\text{and} \quad (-0.948, 0.228, -1.232).$$

By inspection, the minimum point on the surface $f(x, y) = x^3 + y^2 + 2xy$ subject to the constraint $x = y^2 - 1$ is $(-0.948, 0.228, -1.232)$. However, the other two points are neither minimum nor maximum points. Note that as *y* increases in value, so does *x*, and so will $z = f(x, y)$. Thus, an ant walking on this parabolic path on the surface of f can get no lower than $z = -1.232$, but can achieve as high a *z* value as it desires, assuming it walks far enough, all the while staying on the path.

Example 30.4: Consider the portion of the plane $2x + 4y + 5z = 20$ in the first octant. Find the point on the plane closest to the origin.

Solution: The point on the plane closest to the origin will lie on a line orthogonal to the plane. Let (x, y, z) be a point on the plane, so the distance d between this point and the origin (0,0,0) is

$$d(x, y, z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}.$$

However, note that not all variables are independent—they are constrained to one another by the plane's equation. We can isolate one of the variables in the plane. For example, $z = 4 - \frac{2}{5}x - \frac{4}{5}y$. Thus, *d* can be written as a function of *x* and *y* only, and the radicand is expanded:

$$\begin{aligned} d(x, y) &= \sqrt{x^2 + y^2 + \left(4 - \frac{2}{5}x - \frac{4}{5}y\right)^2} \\ &= \sqrt{\frac{29}{25}x^2 + \frac{41}{25}y^2 + \frac{16}{25}xy - \frac{16}{5}x - \frac{32}{5}y + 16}. \end{aligned}$$

Variables *x* and *y* also obey another constraint: both must be non-negative. This will ensure that *z* is also non-negative.

Taking partial derivatives and simplifying, we have

$$d_x = \frac{\frac{29}{25}x + \frac{8}{25}y - \frac{8}{5}}{\sqrt{\frac{29}{25}x^2 + \frac{41}{25}y^2 + \frac{16}{25}xy - \frac{16}{5}x - \frac{32}{5}y + 16}},$$

$$d_y = \frac{\frac{8}{25}x + \frac{41}{25}y - \frac{16}{5}}{\sqrt{\frac{29}{25}x^2 + \frac{41}{25}y^2 + \frac{16}{25}xy - \frac{16}{5}x - \frac{32}{5}y + 16}}.$$

When set to 0, the denominators can be ignored. Thus, only the numerators are considered, and we have

$$\frac{29}{25}x + \frac{8}{25}y - \frac{8}{5} = 0 \qquad \text{and} \qquad \frac{8}{25}x + \frac{41}{25}y - \frac{16}{5} = 0.$$

Placing the constants to the right of the equality and multiplying by 25 to clear fractions, we then have a simplified system in two variables:

$$\begin{aligned} 29x + 8y &= 40 \\ 8x + 41y &= 80. \end{aligned}$$

Using any method (such as elimination) to solve this system, we find that $x = \frac{8}{9}$, $y = \frac{16}{9}$, and substituting these into the plane's equation, we have $z = \frac{20}{9}$. Thus, the point $\left(\frac{8}{9}. \frac{16}{9}, \frac{20}{9}\right)$ is the point on the plane closest to the origin.

We can check this by using normal vectors and techniques used when discussing planes and lines:

Note that a vector orthogonal to the plane is $\mathbf{v} = \langle 2,4,5 \rangle$ and that a line passing through the origin and parallel to this vector (thus, the line is orthogonal to the plane) has the equation

$$\langle 0,0,0 \rangle + t\langle 2,4,5 \rangle = \langle 2t, 4t, 5t \rangle.$$

We determine where this line intersects the plane (see Example 13.4):

$$\begin{aligned} 2(2t) + 4(4t) + 5(5t) &= 20 \\ 4t + 16t + 25t &= 20 \\ 45t &= 20 \\ t &= \frac{4}{9}. \end{aligned}$$

Substituting $t = \frac{4}{9}$ into $\langle 2t, 4t, 5t \rangle$ gives the vector $\langle 2\left(\frac{4}{9}\right), 4\left(\frac{4}{9}\right), 5\left(\frac{4}{9}\right) \rangle = \langle \frac{8}{9}. \frac{16}{9}, \frac{20}{9} \rangle$. Recall that this vector's foot is at the origin, so its head corresponds to the actual point, $\left(\frac{8}{9}. \frac{16}{9}, \frac{20}{9}\right)$.

An alternative method that involves less algebra is shown in Example 32.5.

Example 30.5: Consider the portion of the plane $2x + 4y + 5z = 20$ in the first octant. A rectangular box is situated with one corner at the origin and its opposite corner on the plane so that the box's edges lie along (or are parallel to) the *x*-axis, *y*-axis or *z*-axis. Find the largest possible volume of such a box, keeping the box to within the first octant.

Solution: The volume of a box with edges of length *x*, *y* and *z* is $V(x, y, z) = xyz$. However, as in the previous example, *z* is dependent on *x* and *y* by the equation $z = 4 - \frac{2}{5}x - \frac{4}{5}y$, so that we have

$$V(x, y) = xy\left(4 - \frac{2}{5}x - \frac{4}{5}y\right) = 4xy - \frac{2}{5}x^2y - \frac{4}{5}xy^2.$$

The partial derivatives are

$$V_x = 4y - \frac{4}{5}xy - \frac{4}{5}y^2 \quad \text{and} \quad V_y = 4x - \frac{2}{5}x^2 - \frac{8}{5}xy.$$

These are set equal to zero, and multiplied by 5 to clear fractions. Note that we can factor *y* from V_x and *x* from V_y:

$$\begin{aligned} V_x = 0: &\quad y(20 - 4x - 4y) = 0 \\ V_y = 0: &\quad x(20 - 2x - 8y) = 0. \end{aligned}$$

One solution to this system is *x* = 0 and *y* = 0, but this can be dismissed since it would result in a box with volume 0, a minimum volume, but not a maximum as we seek. Thus, we examine the other factors:

$$\begin{aligned} 20 - 4x - 4y &= 0 \\ 20 - 2x - 8y &= 0. \end{aligned}$$

This system is simplified slightly:

$$\begin{aligned} x + y &= 5 \\ x + 4y &= 10. \end{aligned}$$

The solution is $x = \frac{10}{3}$ and $y = \frac{5}{3}$, so that $z = \frac{4}{3}$. If we so desire, we can use the second derivative test for two-variable functions to show that this gives a maximum volume.

Thus, the largest possible box will have a volume of $\left(\frac{10}{3}\right)\left(\frac{5}{3}\right)\left(\frac{4}{3}\right) = \frac{200}{27} \approx 7.407$ cubic units.

An alternative method is shown in Example 32.6.

31. Constrained Optimization: The Extreme Value Theorem

Suppose a continuous function $z = f(x, y)$ in R^3 has constraints on the independent variables x and y in such a way that the constraint region (when viewed as a region in the *xy*-plane) is a closed and bounded subset of the plane. By *closed*, we mean that the region includes its boundaries, and by *bounded*, the region is of finite area with no asymptotic "tails" trending to infinity.

Under these conditions, it is guaranteed that both absolute minimum and absolute maximum points must exist on the surface representing f subject to the constraints. This is called the **extreme value theorem** (EVT).

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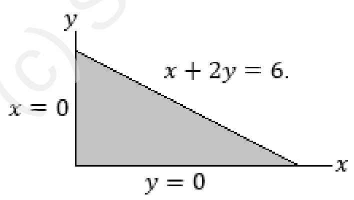
Example 31.1: Find the extreme points on the surface $z = f(x, y) = x^2 + y^2 - 8x - 7y + 3xy$ such that $x \geq 0$, $y \geq 0$ and $x + 2y \leq 6$.

Solution: First, sketch the region in the *xy*-plane defined by the constraints. Recall that $x = 0$ represents the *y*-axis and that $x \geq 0$ suggests to shade to the right of the *y*-axis. Use similar logic when sketching the other two boundary lines. Note that we have a triangle. It is closed (includes its boundaries) and bounded (of finite area).



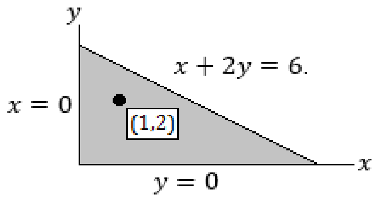
We check for possible critical points contained within the region. Using routine optimization techniques, we differentiate with respect to *x* and with respect to *y*:

$$f_x(x, y) = 2x - 8 + 3y$$
$$f_y(x, y) = 2y - 7 + 3x.$$

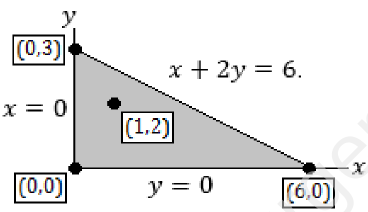
Setting these equations equal to 0, we solve a system:

$$\begin{aligned} 2x - 8 + 3y &= 0 \\ 2y - 7 + 3x &= 0, \end{aligned} \quad \text{which simplifies to} \quad \begin{aligned} 2x + 3y &= 8 \\ 3x + 2y &= 7. \end{aligned}$$

The solution of this system is $x = 1$ and $y = 2$. Note that this point satisfies all three constraints simultaneously. This is a valid critical point.



Next, we identify all vertices (corners) of the region. These will be critical points too.



Then we check for critical points along each boundary, one at a time.

• For $x = 0$ (the y-axis), where $0 \le y \le 3$, we substitute into the function:

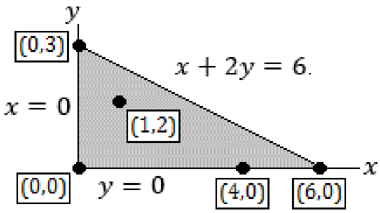
$$f(0,y) = (0)^2 + y^2 - 8(0) - 7y + 3(0)y$$
$$f(y) = y^2 - 7y. \quad \text{(simplified)}$$

Differentiating, we have $f'(y) = 2y - 7$, and when set equal to 0, we find that $y = \frac{7}{2}$. However, this value is outside the range $0 \le y \le 3$, so it is ignored.

• For $y = 0$ (the x-axis), where $0 \le x \le 6$, we substitute into the function:

$$f(x,0) = x^2 + (0)^2 - 8x - 7(0) + 3x(0)$$
$$f(x) = x^2 - 8x. \quad \text{(simplified)}$$

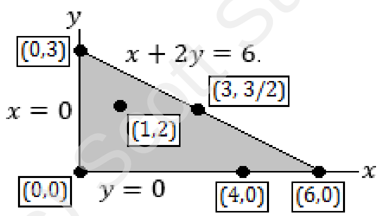
Differentiating, we have $f'(x) = 2x - 8$, and when set equal to 0, we find that $x = 4$ is a critical value. This is inside the range $0 \le x \le 6$, so it is included.



• For $x + 2y = 6$, we solve for a convenient variable and substitute. Let's use $x = 6 - 2y$.

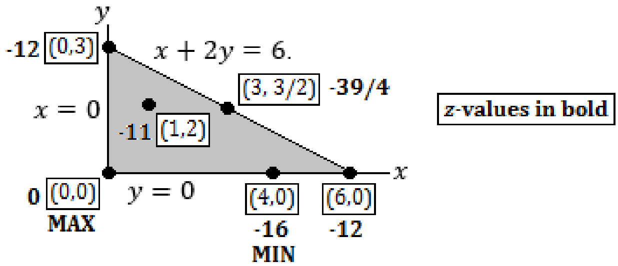
$$f(6-2y, y) = (6-2y)^2 + y^2 - 8(6-2y) - 7y - 3(6-2y)y$$
$$f(y) = 36 - 24y + 4y^2 + y^2 - 48 + 16y - 7y - 18y + 6y^2$$
$$f(y) = 11y^2 - 33y - 12. \text{ (simplified)}$$

Differentiating, we have $f'(y) = 22y - 33$, and when set equal to 0, we find that $y = \frac{3}{2}$ is a critical value. Since $x = 6 - 2y$, we have $x = 6 - 2\left(\frac{3}{2}\right) = 3$. These values are within the respective ranges for the x and y variables, so this is also a critical point.



We have six critical points. These are evaluated into the function:

$$z = f(0,0) = (0)^2 + (0)^2 - 8(0) - 7(0) + 3(0)(0) = 0,$$
$$z = f(4,0) = (4)^2 + (0)^2 - 8(4) - 7(0) + 3(4)(0) = -16,$$
$$z = f(6,0) = (6)^2 + (0)^2 - 8(6) - 7(0) + 3(6)(0) = -12,$$
$$z = f(0,3) = (0)^2 + (3)^2 - 8(0) - 7(3) + 3(0)(3) = -12,$$
$$z = f(1,2) = (1)^2 + (2)^2 - 8(1) - 7(2) + 3(1)(2) = -11,$$
$$z = f(3,3/2) = (3)^2 + (3/2)^2 - 8(3) - 7(3/2) + 3(3)(3/2) = -39/4.$$



By inspection, the absolute maximum value on the surface of f subject to the constraints is $z = 0$, and it occurs at (0,0,0), the absolute maximum point. The absolute minimum value is $z = -16$ and occurs at the absolute minimum point $(4,0,-16)$. The other points are then ignored.

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In the previous example and the one that follows, it is important to remember that we are searching for highest and lowest points on a surface. In the images in each example, we tend to concentrate on the projection of the region onto the *xy*-plane and track our calculations on this projection. However, if these regions are projected back to the surface, it will conform to the surface. In a sense, it is similar to fencing that encloses a yard on hilly terrain. The extreme points may lie within the yard, along one of the fences, or at a corner of fences.

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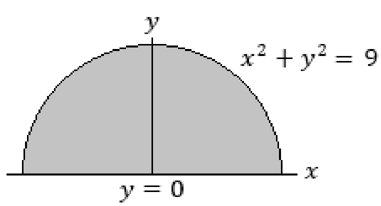
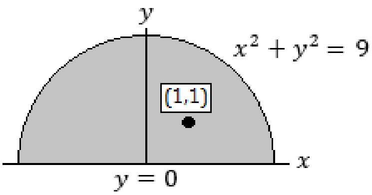
Example 31.2: Find the extreme points on the surface $z = f(x,y) = (x-1)^2 + (y-1)^2$ such that $y \geq 0$ and $x^2 + y^2 \leq 9$.

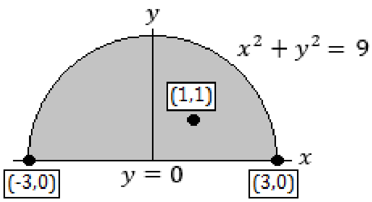
Solution: We sketch the constraint region in the *xy*-plane:



Note that $f(x,y) = (x-1)^2 + (y-1)^2$ is the paraboloid $z = x^2 + y^2$ that has been shifted one unit in the positive x direction and one unit in the positive y direction. That is, its vertex is at (1,1,0). Note that when $x = 1$ and $y = 1$, it is contained within the region as defined by the constraints. This is a critical point.



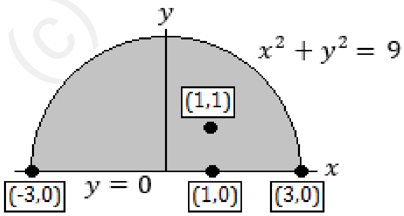
The points at which the circle meets the *x*-axis are critical points too.



Along the *x*-axis, we let *y* = 0, noting that $-3 \leq x \leq 3$, and substitute into the function f:

$$f(x, 0) = (x-1)^2 + (0-1)^2$$
$$f(x) = x^2 - 2x + 2. \text{ (simplified)}$$

Differentiating, we have $f'(x) = 2x - 2$, and setting this equal to 0, we have that $x = 1$. This is within the bounds of x, so it is a critical value, while (1,0) is a critical point.



For the circular boundary, we rewrite it as $y = \sqrt{9 - x^2}$, with $-3 \leq x \leq 3$, noting that we need only the positive root. This is substituted into f:

$$f\left(x, \sqrt{9-x^2}\right) = (x-1)^2 + \left(\sqrt{9-x^2} - 1\right)^2$$
$$f(x) = x^2 - 2x + 1 + 9 - x^2 - 2\sqrt{9-x^2} + 1$$
$$f(x) = -2x + 11 - 2\sqrt{9-x^2}.$$

Differentiating, we have

$$f'(x) = -2 + \frac{2x}{\sqrt{9-x^2}}, \quad \text{for } -3 < x < 3.$$

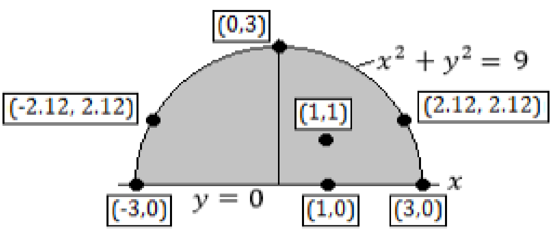
This is set equal to 0, and simplified:

$$\begin{aligned} -2 + \frac{2x}{\sqrt{9-x^2}} &= 0 \\ \frac{2x}{\sqrt{9-x^2}} &= 2 \\ x &= \sqrt{9-x^2} \\ x^2 &= \left(\sqrt{9-x^2}\right)^2 \\ x^2 &= (9-x^2) \\ 2x^2 &= 9 \\ x^2 &= \frac{9}{2}. \end{aligned}$$

(Note that the values $x = \pm 3$ have already been accounted for previously.)

Thus, we have $x = 3/\sqrt{2}$ and $x = -3/\sqrt{2}$. These are approximately $x = \pm 2.12$. They fall within the interval $-3 < x < 3$. Since $y = \sqrt{9-x^2}$ along the boundary, when $x = 3/\sqrt{2}$, we have $y = \sqrt{9 - \left(3/\sqrt{2}\right)^2} = \sqrt{9-(9/2)} = \sqrt{9/2} = 3/\sqrt{2}$. Similarly, when $x = -3/\sqrt{2}$, then $y = 3/\sqrt{2}$. These are also critical points.

We also consider the point (0,3), since the variable *y* will be constrained within the interval $0 \leq y \leq 3$. Thus, we have a total of seven critical points.



These are evaluated into the function:

$$z = f(1{,}0) = \left((1) - 1\right)^2 + \left((0) - 1\right)^2 = 1,$$
$$z = f(3{,}0) = \left((3) - 1\right)^2 + \left((0) - 1\right)^2 = 5,$$
$$z = f(-3{,}0) = \left((-3) - 1\right)^2 + \left((0) - 1\right)^2 = 17,$$
$$z = f(1{,}1) = \left((1) - 1\right)^2 + \left((1) - 1\right)^2 = 0,$$
$$z = f\left(-3/\sqrt{2}, 3/\sqrt{2}\right) = \left(\left(-3/\sqrt{2}\right) - 1\right)^2 + \left(\left(3/\sqrt{2}\right) - 1\right)^2 \approx 11,$$
$$z = f\left(3/\sqrt{2}, 3/\sqrt{2}\right) = \left(\left(3/\sqrt{2}\right) - 1\right)^2 + \left(\left(3/\sqrt{2}\right) - 1\right)^2 \approx 2.515,$$
$$z = f(0{,}3) = \left((0) - 1\right)^2 + \left((3) - 1\right)^2 = 5.$$

The absolute maximum point is $(-3{,}0{,}17)$, and the absolute minimum point is $(1{,}1{,}0)$. The rest are ignored.

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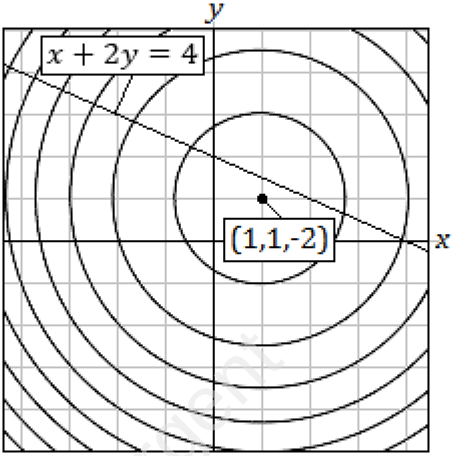
32. Method of Lagrange Multipliers

The Method of Lagrange Multipliers is a generalized approach to solving constrained optimization problems. Assume that we are seeking to optimize a function $z = f(x, y)$ subject to a "path" constraint defined implicitly by $g(x, y) = c$. The process usually follows these steps:

1. Define a function $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$.
2. Find the partial derivatives L_x, L_y and L_λ. Note that $L_\lambda = -g(x, y) + c$.
3. Set these partial derivatives to 0. Note that $L_\lambda = 0$ is the same as $g(x, y) = c$, the original path constraint. Also note any restrictions on *x* and *y*, as it may be necessary to consider locations where the derivative fails to exist.
4. Isolate the λ in the equations $L_x = 0$ and $L_y = 0$, then equate the two expressions. This will "drop out" the λ, leaving an equation in *x* and *y* only. If possible, isolate *x* or *y*.
5. Substitute the result from step 4 into the equation $g(x, y) = c$, which will now be a single-variable equation. Solve for the remaining variable.
6. Back substitute to find corresponding values for the other variable, and for *z*.
7. Compare *z* values. The smallest will be a minimum, the largest a maximum. If there is just one *z* value, then other observations, such as cross-sections, may be needed to determine whether the point is a minimum or maximum.

The following examples illustrate possible situations that may occur.

Example 32.1: Find the minimum value of $z = f(x, y) = x^2 + y^2 - 2x - 2y$ subject to the constraint $x + 2y = 4$.



Solution: To the right is a contour map in R^2 of the surface defined by f, and the constraint $x + 2y = 4$ shown as a line. The actual surface is a paraboloid that opens up and has a minimum point at $(1,1,-2)$, its vertex. The path, when conformed to the surface, is a cross-section of the paraboloid, itself a parabola. Thus, by inspecting the geometry of the problem, the extreme point on this path/parabola will be a minimum point.

Note that the path is slightly off-set from the vertex. It is reasonable to assume that the lowest point on the constraint path will be near the vertex, but clearly cannot be at the paraboloid's vertex.

First, create a new function L, clearing parentheses at the end:

$$
\begin{aligned}
L(x, y, \lambda) &= f(x, y) - \lambda(g(x, y) - c) \\
L(x, y, \lambda) &= x^2 + y^2 - 2x - 2y - \lambda(x + 2y - 4) \\
L(x, y, \lambda) &= x^2 + y^2 - 2x - 2y - \lambda x - 2\lambda y + 4\lambda.
\end{aligned}
$$

Next, find its partial derivatives:

$$
\begin{aligned}
L_x &= 2x - 2 - \lambda \\
L_y &= 2y - 2 - 2\lambda \\
L_\lambda &= -x - 2y + 4.
\end{aligned}
$$

Now, set these to 0:

$$
\begin{aligned}
2x - 2 - \lambda = 0 \quad & (\mathbf{1}) \\
2y - 2 - 2\lambda = 0 \quad & (\mathbf{2}) \\
-x - 2y + 4 = 0. \quad & (\mathbf{3})
\end{aligned}
$$

In equations **(1)** and **(2)**, isolate the λ:

$$\lambda = 2x - 2 \quad \text{and} \quad \lambda = y - 1.$$

There are no restrictions on x or y. Now, equate and simplify. Note that λ is no longer present.

$$2x - 2 = y - 1, \quad \text{which gives} \quad y = 2x - 1.$$

Note that equation **(3)** from above is the same as the constraint $x + 2y = 4$. Substitute the equation $y = 2x - 1$ into the simplified form of equation **(3)**, and solve for x:

$$\begin{aligned} x + 2(2x - 1) &= 4 \\ 5x - 2 &= 4 \\ 5x &= 6 \\ x &= \frac{6}{5}. \end{aligned}$$

Find y by substituting $x = \frac{6}{5}$ into the equation $y = 2x - 1$:

$$y = 2\left(\frac{6}{5}\right) - 1 = \frac{7}{5}.$$

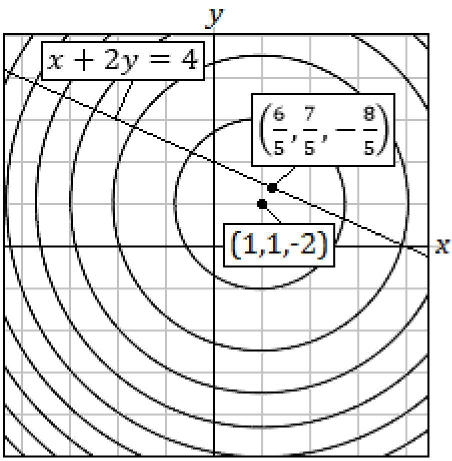
Lastly, find z using the original function f:

$$f\left(\frac{6}{5}, \frac{7}{5}\right) = \left(\frac{6}{5}\right)^2 + \left(\frac{7}{5}\right)^2 - 2\left(\frac{6}{5}\right) - 2\left(\frac{7}{5}\right) = -\frac{8}{5}.$$

The minimum point of $z = f(x, y) = x^2 + y^2 - 2x - 2y$ subject to the constraint $x + 2y = 4$ is

$$\left(\frac{6}{5}, \frac{7}{5}, -\frac{8}{5}\right).$$

This seems to agree with our assumption that it would be "close" to the surface's minimum at $(1,1,-2)$, its component values each a little higher than those of the vertex.



Graph for Example 32.1

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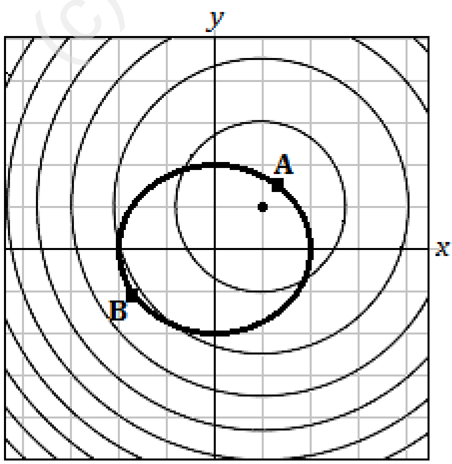
Example 32.2: Let $z = f(x, y) = x^2 + y^2 - 2x - 2y$. Find the minimum value of f subject to the constraint $x^2 + y^2 = 4$.

Solution: This is the same surface as in the previous example. However, the constraint path is a circle of radius 2 (as viewed on the *xy*-plane). When conformed to the surface f, the path will rise and fall along with the surface. Observing the path (in bold) in relation to the contours, we can estimate where the path's lowest point may be, and where its highest point may be:



A: Probable lowest point
B: Probable highest point

Using the Method of Lagrange Multipliers, we start by building function L:

$$L(x,y,\lambda) = x^2 + y^2 - 2x - 2y - \lambda(x^2 + y^2 - 4)$$
$$L(x,y,\lambda) = x^2 + y^2 - 2x - 2y - \lambda x^2 - \lambda y^2 + 4\lambda. \quad \text{(Simplified)}$$

Now find the partial derivatives:

$$L_x = 2x - 2 - 2\lambda x$$
$$L_y = 2y - 2 - 2\lambda y$$
$$L_\lambda = -x^2 - y^2 + 4.$$

Set each partial derivative to 0. Again, note that $L_\lambda = 0$ (Equation **(3)**) is the constraint path:

$$2x - 2 - 2\lambda x = 0 \quad (\mathbf{1})$$
$$2y - 2 - 2\lambda y = 0 \quad (\mathbf{2})$$
$$x^2 + y^2 = 4. \quad (\mathbf{3})$$

Isolate λ in equations **(1)** and **(2)**, then equate. Note any restrictions on the variables:

$$\lambda = \frac{x-1}{x} \quad \text{and} \quad \lambda = \frac{y-1}{y}, \quad \text{so that} \quad \frac{x-1}{x} = \frac{y-1}{y} \quad (x, y \neq 0).$$

Clearing fractions, we have

$$y(x-1) = x(y-1)$$
$$xy - y = xy - x$$
$$y = x.$$

Substitute $y = x$ into **(3)**:

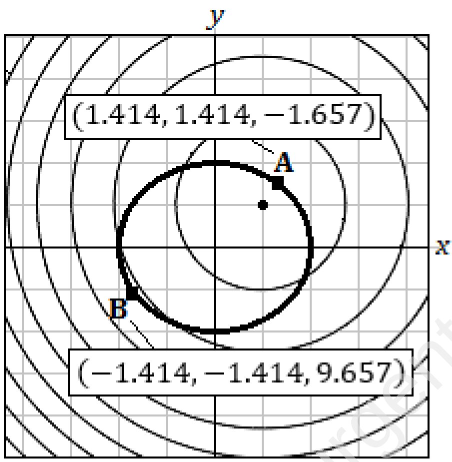
$$x^2 + x^2 = 4$$
$$2x^2 = 4$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$

Since $y = x$, we have $y = \sqrt{2}$ when $x = \sqrt{2}$, and $y = -\sqrt{2}$ when $x = -\sqrt{2}$. There are two critical points:

$$\left(\sqrt{2}, \sqrt{2}, f\left(\sqrt{2}, \sqrt{2}\right)\right) \quad \text{and} \quad \left(-\sqrt{2}, -\sqrt{2}, f\left(-\sqrt{2}, -\sqrt{2}\right)\right).$$

We now evaluate the function at each of these x and y values:

$$f(\sqrt{2},\sqrt{2}) = (\sqrt{2})^2 + (\sqrt{2})^2 - 2\sqrt{2} - 2\sqrt{2} = 4 - 4\sqrt{2} \approx -1.657,$$
$$f(-\sqrt{2},-\sqrt{2}) = (-\sqrt{2})^2 + (-\sqrt{2})^2 + 2\sqrt{2} + 2\sqrt{2} = 4 + 4\sqrt{2} \approx 9.657.$$



By observation,

$$\left(\sqrt{2},\sqrt{2},f\left(\sqrt{2},\sqrt{2}\right)\right) \approx (1.414, 1.414, -1.657)$$

is the minimum point (**A**) on the surface subject to the constraint, while

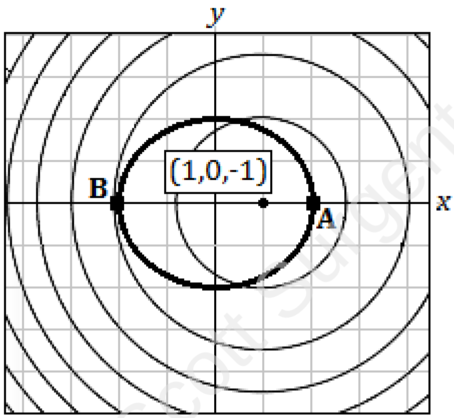
$$\left(-\sqrt{2},-\sqrt{2},f\left(-\sqrt{2},-\sqrt{2}\right)\right) \approx (-1.414, -1.414, 9.657)$$

is the maximum point (**B**) on the surface subject to the constraint. We also see that these points are where we surmised they would be: the minimum point on the path is closest to the minimum point of the entire surface, while the maximum point is farthest away.

The restrictions that $x \neq 0$ or $y \neq 0$ ultimately did not play a role in this example. In the next example, it does.

Example 32.3: Let $z = f(x, y) = x^2 + y^2 - 2x$. Find the minimum and maximum values of f subject to the constraint $x^2 + y^2 = 4$.

Solution: The surface as defined by f is a paraboloid with vertex at $(1,0,-1)$. Since the paraboloid opens upward, the vertex is the absolute minimum point on the surface. We show the contour map and identify the path constraint (in bold), which is the circle of radius 2, centered at the origin. It is reasonable to infer that the minimum point on the surface subject to the constraint is probably the point closest to the vertex (denoted **A**), and the maximum point is farthest away from the vertex (denoted **B**) given that the surface rises the farther away one moves from the origin.



We build function L:

$$L(x, y, \lambda) = x^2 + y^2 - 2x - \lambda x^2 - \lambda y^2 + 4\lambda.$$

Now find the partial derivatives:

$$\begin{aligned} L_x &= 2x - 2 - 2\lambda x \\ L_y &= 2y - 2\lambda y \\ L_\lambda &= -x^2 - y^2 + 4. \end{aligned}$$

Set each partial derivative to 0:

$$\begin{aligned} 2x - 2 - 2\lambda x &= 0 \quad (\mathbf{1}) \\ 2y - 2\lambda y &= 0 \quad (\mathbf{2}) \\ x^2 + y^2 &= 4. \quad (\mathbf{3}) \end{aligned}$$

Isolate λ in equations **(1)** and **(2)**, then equate. Note any restrictions on the variables:

$$\lambda = \frac{x-1}{x} \quad \text{and} \quad \lambda = \frac{y}{y} = 1, \quad \text{so that} \quad \frac{x-1}{x} = 1 \quad (x, y \neq 0)$$

Simplifying $\frac{x-1}{x} = 1$, we get $x - 1 = x$, or $0 = -1$, which is a false statement. It seems the process has stalled. However, it has not. The nature of the algebra in this step forces $x \neq 0$ and $y \neq 0$, but in truth, the surface and the constraint are defined when $x = 0$ or $y = 0$. In equation **(2)**, which is $2y - 2\lambda y = 0$, note that $y = 0$ is also a solution.

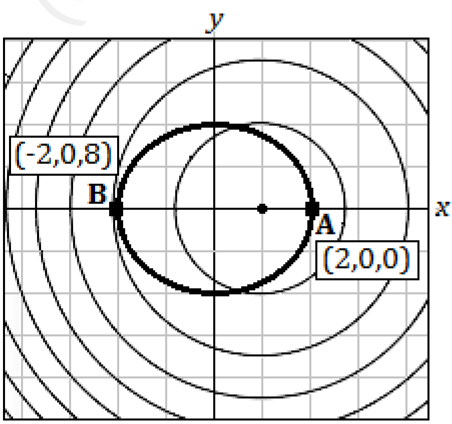
Substituting this into equation **(3)**, the original constraint, we can solve for x:

$$\begin{aligned} x^2 + 0^2 &= 4 \\ x^2 &= 4 \\ x &= \pm 2. \end{aligned}$$

Thus,we have two critical points, $\big(2,0,f(2,0)\big)$ and $\big(-2,0,f(-2,0)\big)$. The z values are

$$f(2,0) = 2^2 + 0^2 - 2(2) = 0 \quad \& \quad f(-2,0) = (-2)^2 + 0^2 - 2(-2) = 8.$$

The point $(2,0,0)$ is the minimum point (**A**) on the surface subject to the constraint, and the point $(-2,0,8)$ is the maximum point (**B**) on the surface subject to the constraint. This agrees with our original intuition.

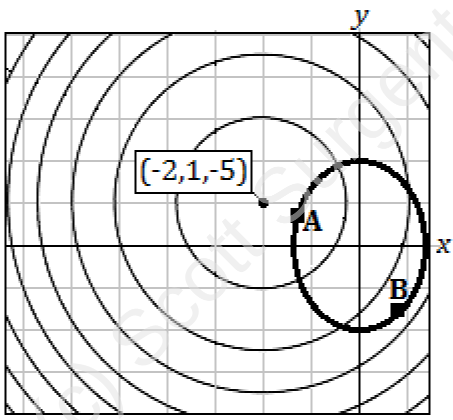


Graph for example 32.3

The previous three examples have been efficient, in that the algebra has not been too difficult. In the next example, we encounter a situation where the algebra may pose a challenge.

Example 32.4: Let $z = f(x,y) = x^2 + y^2 + 4x - 2y$. Find the minimum and maximum values of f subject to the constraint $2x^2 + y^2 = 4$.

Solution: The surface is a parabolid opening upward. Its vertex, $(-2,1,-5)$, is the absolute minimum point on this surface. The path is an ellipse centered at the origin with a major axis of 4 units in the *y* direction (± 2 units from the origin) and a minor axis of $2\sqrt{2}$ units in the *x* direction ($\pm\sqrt{2}$ units from the origin). We label what we think may be the location of the minimum point (**A**) of the surface subject to the constrain, and what we think may be the maximum point (**B**) of the surface, subject to the constraint.



We follow the same steps as before:

$$L(x,y,\lambda) = x^2 + y^2 + 4x - 2y - 2\lambda x^2 - \lambda y^2 + 4\lambda.$$

The partial derivatives are

$$\begin{aligned} L_x &= 2x + 4 - 4\lambda x \\ L_y &= 2y - 2 - 2\lambda y \\ L_\lambda &= -2x^2 - y^2 + 4. \end{aligned}$$

Setting each to 0, we have a system:

$$\begin{aligned} 2x + 4 - 4\lambda x &= 0 \quad (\mathbf{1}) \\ 2y - 2 - 2\lambda y &= 0 \quad (\mathbf{2}) \\ 2x^2 + y^2 &= 4. \quad (\mathbf{3}) \end{aligned}$$

Isolate λ in equations **(1)** and **(2)**, then equate. Note any restrictions on the variables:

$$\lambda = \frac{x+2}{2x} \quad \text{and} \quad \lambda = \frac{y-1}{y}, \quad \text{so that} \quad \frac{x+2}{2x} = \frac{y-1}{y} \quad (x, y \neq 0)$$

Clearing fractions, we have

$$\begin{aligned} y(x+2) &= 2x(y-1) \\ xy + 2y &= 2xy - 2x \\ 2y - xy &= -2x \\ y(2-x) &= -2x \\ y &= \frac{2x}{x-2} \qquad (x \neq 2). \end{aligned}$$

Substitute this into **(3)**:

$$2x^2 + \left(\frac{2x}{x-2}\right)^2 = 4.$$

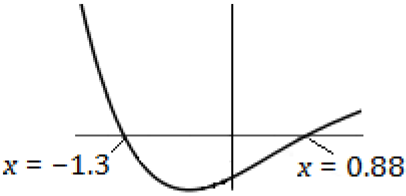
Clear fractions:

$$(x-2)^2 2x^2 + (2x)^2 = 4(x-2)^2.$$

Expanding by multiplication and collecting terms, we have

$$x^4 - 4x^3 + 4x^2 + 8x - 8 = 0.$$

It is difficult to isolate x in a quartic polynomial. Instead, the roots are determined graphically. The roots are $x \approx -1.3$ and $x \approx 0.88$:



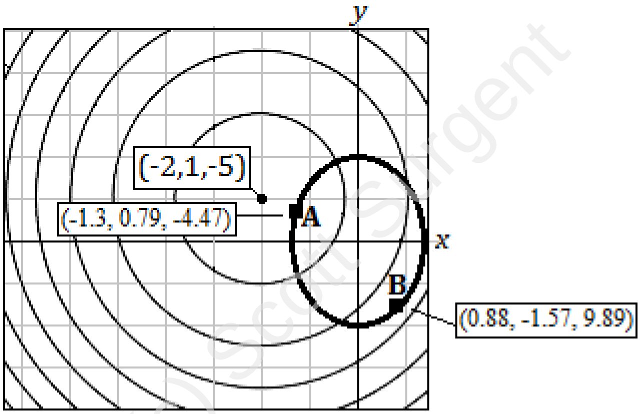
Now use the equation $y = \frac{2x}{x-2}$ to determine y at each x-value:

$$y = \frac{2(-1.3)}{(-1.3)-2} \approx 0.79 \quad \text{and} \quad y = \frac{2(0.88)}{(0.88)-2} \approx -1.57.$$

We then find the z-values:

$$z = f(-1.3, 0.79) = (-1.3)^2 + (0.79)^2 + 4(-1.3) - 2(0.79) \approx -4.47,$$
$$z = f(0.88, -1.57) = (0.88)^2 + (-1.57)^2 + 4(0.88) - 2(-1.57) \approx 9.89.$$

Thus, the point $(-1.3, 0.79, -4.47)$ is the minimum point (**A**) on the surface subject to the constraint, and the point $(0.88, -1.57, 9.89)$ is the maximum point (**B**) on the surface subject to the constraint.



The restrictions imposed on *x* and *y* during the algebra steps did not play a role in finding the solutions.

Lagrange Multiplies can be extended into situations with three or more variables.

Example 32.5: Consider the portion of the plane $2x + 4y + 5z = 20$ in the first octant. Find the point on the plane closest to the origin. (This is the same as Example 30.4)

Solution: If (x, y, z) is a point on the plane, then its distance from the origin is $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Using the constraint $2x + 4y + 5z - 20 = 0$, we build the function L:

$$L(x, y, z, \lambda) = \sqrt{x^2 + y^2 + z^2} - \lambda(2x + 4y + 5z - 20).$$

Then we find partial derivatives:

$$L_x = \frac{x}{\sqrt{x^2+y^2+z^2}} - 2\lambda,$$
$$L_y = \frac{y}{\sqrt{x^2+y^2+z^2}} - 4\lambda,$$
$$L_z = \frac{z}{\sqrt{x^2+y^2+z^2}} - 5\lambda,$$
$$L_\lambda = -2x - 4y - 5z + 20.$$

These are then set equal to 0:

$$\frac{x}{\sqrt{x^2+y^2+z^2}} - 2\lambda = 0, \qquad \text{so that} \qquad \lambda = \frac{x}{2\sqrt{x^2+y^2+z^2}}, \qquad \textbf{(1)}$$
$$\frac{y}{\sqrt{x^2+y^2+z^2}} - 4\lambda = 0, \qquad \text{so that} \qquad \lambda = \frac{y}{4\sqrt{x^2+y^2+z^2}}, \qquad \textbf{(2)}$$
$$\frac{z}{\sqrt{x^2+y^2+z^2}} - 5\lambda = 0, \qquad \text{so that} \qquad \lambda = \frac{z}{5\sqrt{x^2+y^2+z^2}}, \qquad \textbf{(3)}$$
$$-2x - 4y - 5z + 20 = 0, \qquad \text{so that} \qquad 2x + 4y + 5z = 20. \qquad \textbf{(4)}$$

Equating **(1)** and **(2)**, we have

$$\frac{x}{2\sqrt{x^2+y^2+z^2}} = \frac{y}{4\sqrt{x^2+y^2+z^2}}.$$

Clearing fractions, we have $y = 2x$. Then, equating **(1)** and **(3)** and clearing fractions, we have $z = \frac{5}{2}x$. These are substituted into equation **(4)**:

$$2x + 4(2x) + 5\left(\frac{5}{2}x\right) = 20.$$

Simplifying, we have $\frac{45}{2}x = 20$, so that $x = \frac{40}{45} = \frac{8}{9}$. Since $y = 2x$, we have $y = 2\left(\frac{8}{9}\right) = \frac{16}{9}$, and since $z = \frac{5}{2}x$, we have $z = \frac{5}{2}\left(\frac{8}{9}\right) = \frac{40}{18} = \frac{20}{9}$.

The point on the plane $2x + 4y + 5z = 20$ closest to the origin is $\left(\frac{8}{9}, \frac{16}{9}, \frac{20}{9}\right)$.

Example 32.6: Consider the portion of the plane $2x + 4y + 5z = 20$ in the first octant. A rectangular box is situated with one corner at the origin and its opposite corner on the plane so that the box's edges lie along (or are parallel to) the x-axis, y-axis or z-axis. Find the largest possible volume of such a box, keeping the box to within the first octant. (This is the same as Example 30.5)

Solution: The volume of the box is given by $V(x, y, z) = xyz$, and along with the constraint $2x + 4y + 5z - 20 = 0$, we build function L:

$$L(x, y, z, \lambda) = xyz - \lambda(2x + 4y + 5z - 20).$$

Taking partial derivatives, we have

$$\begin{aligned} L_x &= yz - 2\lambda, \\ L_y &= xz - 4\lambda, \\ L_z &= xy - 5\lambda, \\ L_\lambda &= -2x - 4y - 5z + 20. \end{aligned}$$

Setting each to 0, we have

$$yz - 2\lambda = 0, \quad \text{so that} \quad \lambda = \frac{yz}{2}, \qquad (\mathbf{1})$$

$$xz - 4\lambda = 0, \quad \text{so that} \quad \lambda = \frac{xz}{4}, \qquad (\mathbf{2})$$

$$xy - 5\lambda = 0, \quad \text{so that} \quad \lambda = \frac{xy}{5}, \qquad (\mathbf{3})$$

$$-2x - 4y - 5z + 20 = 0, \quad \text{so that} \quad 2x + 4y + 5z = 20. \qquad (\mathbf{4})$$

Equating **(1)** and **(2)**, we have

$$\frac{yz}{2} = \frac{xz}{4}, \quad \text{so that} \quad y = \frac{1}{2}x.$$

Equating **(1)** and **(3)**, we have

$$\frac{yz}{2} = \frac{xy}{5}, \quad \text{so that} \quad z = \frac{2}{5}x.$$

These are substituted into **(4)** and variable *x* is isolated:

$$\begin{aligned} 2x+4\left(\frac{1}{2}x\right)+5\left(\frac{2}{5}x\right) &= 20 \\ 2x+2x+2x &= 20 \\ 6x &= 20 \\ x &= \frac{10}{3}. \end{aligned}$$

Since $y=\frac{1}{2}x$, we have $y=\frac{1}{2}\left(\frac{10}{3}\right)=\frac{5}{3}$, and since $z=\frac{2}{5}x$, we have $z=\frac{2}{5}\left(\frac{10}{3}\right)=\frac{4}{3}$. Thus, the dimensions of the largest box will be $\frac{10}{3}\times\frac{5}{3}\times\frac{4}{3}$, with the volume $\left(\frac{10}{3}\right)\left(\frac{5}{3}\right)\left(\frac{4}{3}\right)=\frac{200}{27}$.