29. Unconstrained Optimization

Optimization is the process of determining the highest (maximal) and lowest (minimal) points on a graph. Maximum and minimum points are collectively called *extreme points*, or *extrema*.

Let $z = f(x, y)$ be a function in R^3, and assume that f exists and is continuous over the entire *xy*-plane. That is, its domain is R^2, there being no restrictions on variables *x* and y.

A **critical point** $(x_c y_c, z_c)$, where $z_c = f(x_c, y_c)$, is a point where $f_x(x_c, y_c) = 0$ or does not exist, and where $f_y(x_c, y_c) = 0$ or does not exist. These are the possible extreme points. All minimum and maximum points are **local** (or **relative**), meaning that the point is the lowest or highest point within some interval that includes the point. If it is the lowest or highest point over the entire domain, then the point is an **absolute** minimum or maximum.

The second derivative test for R^2 is one way to determine if a critical point $(x_c y_c, z_c)$ is a minimum, a maximum, or neither. The formula is

$$D = f_{xx}(x_c, y_c) f_{yy}(x_c, y_c) - \left(f_{xy}(x_c, y_c)\right)^2.$$

- If $D > 0$ and if $f_{xx}(x_c, y_c) > 0$, then the graph of f is concave upward, and (x_c, y_c, z_c) is a relative minimum.
- If $D > 0$ and if $f_{xx}(x_c, y_c) < 0$, then the graph of f is concave downward, and (x_c, y_c, z_c) is a relative maximum.
- If $D < 0$, then (x_c, y_c, z_c) is not a minimum nor a maximum. It is a saddle point.
- If $D = 0$, then no conclusion about (x_c, y_c, z_c) can be inferred. Other methods may need to be used to classify the critical point.

When $D > 0$, this forces the signs of $f_{xx}(x_c, y_c)$ and $f_{yy}(x_c, y_c)$ to be the same. Thus, it is sufficient to note the sign of one, since the sign of the other will be the same.

When there are no restrictions on the domain, this process is called **unconstrained optimization**.

♦ • ♦ • ♦ • ♦ • ♦

Example 29.1: Find the critical points of $z = f(x, y) = x^2 + y^2 + 6x - 4y + 2$, and classify these points as minima, maxima or saddle.

Solution: We find the first partial derivatives:

$$f_x(x, y) = 2x + 6$$
$$f_y(x, y) = 2y - 4.$$

Note that the derivatives (as well as the function itself) are defined for all x and all y in R^2. Thus, there are no possible locations where the derivatives "do not exist". We set the partial derivatives to 0, and solve:

$$\begin{array}{c} 2x+6=0 \\ 2y-4=0 \end{array} \quad \text{which gives} \quad \begin{array}{c} x=-3 \\ y=2. \end{array}$$

Thus, we have one critical point, $(-3,2,f(-3,2))$, where $f(-3,2)=-11$. To classify this critical point, we use the second derivative test. The second derivatives are found first (recall that $f_{xy}(x,y)=f_{yx}(x,y)$):

$$f_{xx}(x,y)=2, \qquad f_{yy}(x,y)=2 \quad \text{and} \quad f_{xy}(x,y)=0.$$

By the second derivative test, we have

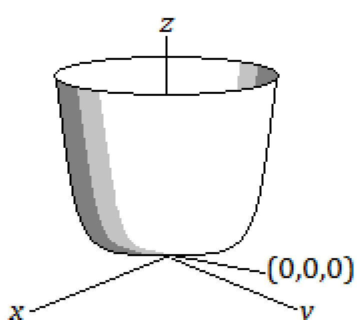
$$\begin{aligned} D &= f_{xx}(-3,2)f_{yy}(-3,2)-\left(f_{xy}(-3,2)\right)^2 \\ &= (2)(2)-0^2 \\ &= 4. \end{aligned}$$

Note that $D>0$ and that $f_{xx}>0$. Therefore, $(-3,2,-11)$ is a local minimum point. The graph of $z=f(x,y)=x^2+y^2+6x-4y+2$ is a paraboloid that opens upward (in the direction of positive z). Its vertex is $(-3,2,-11)$. We conclude that this point is also the absolute minimum point over the entire domain.

♦ • ♦ • ♦ • ♦ • ♦

Example 29.2: Find the critical points of $z=g(x,y)=x^4+y^4$, and classify these points as minima, maxima or saddle.

Solution: The first partial derivatives are $g_x(x,y)=4x^3$ and $g_y(x,y)=4y^3$. Setting each to 0, we get $x=0$ and $y=0$. Note that $z=g(0,0)=0$, so that $(0,0,0)$ is the lone critical point.



$g(x,y)=x^4+y^4$

The second derivatives are $g_{xx}(x,y) = 12x^2$, $g_{yy}(x,y) = 12y^2$ and $g_{xy}(x,y) = 0$. Using the second derivative test, we have

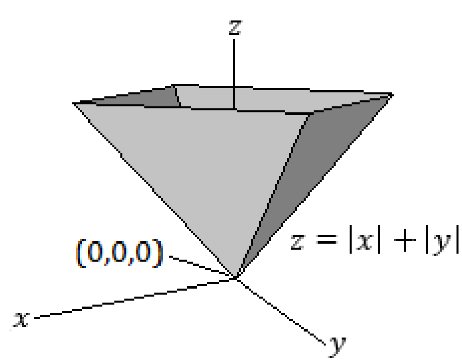
$$\begin{aligned} D &= g_{xx}(0,0)g_{yy}(0,0) - \left(g_{xy}(0,0)\right)^2 \\ &= [12(0)^2][12(0)^2] - 0 \\ &= 0. \end{aligned}$$

The second derivative test yields no useful information. However, note that the cross sections of this surface are $z = x^4$ (when y = 0) and $z = y^4$ (when x = 0). In each case, the point (0,0) is a minimum, so we can infer that (0,0,0) is a local minimum point on the surface of $z = x^4 + y^4$. The surface is bowl-shaped, with a flattened bottom, where (0,0,0) is its vertex. Viewing its graph suggests that the point is the absolute minimum, too.

♦ • ♦ • ♦ • ♦ • ♦

Example 29.3: Find the critical points of $z = h(x,y) = |x| + |y|$, and classify these points as minima, maxima or saddle.

Solution: Since $|x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$, then by inspection, $\frac{d}{dx}|x| = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$, where the derivative is not defined (does not exist) at x = 0. A similar argument shows that for $|y|$, the derivative does not exist at y = 0. Therefore, the point (0,0,0) is a critical point. However, the second derivative test is not applicable. Instead, we can classify the critical point by observing the graph of h, where we see that (0,0,0) is a local and absolute minimum point.



♦ • ♦ • ♦ • ♦ • ♦

Example 29.4: Find the critical points of $z = f(x, y) = x^3 + y^3 - 3x - 27y + 7$, and classify these points as minima, maxima or neither.

Solution: We find the partial derivatives:

$$f_x = 3x^2 - 3$$
$$f_y = 3y^2 - 27.$$

These are set equal to 0 and solved for the variable:

$$\begin{matrix} 3x^2 - 3 = 0 \\ 3y^2 - 27 = 0 \end{matrix}, \quad \text{which simplifies as} \quad \begin{matrix} 3(x^2 - 1) = 0 \\ 3(y^2 - 9) = 0. \end{matrix}$$

From the first equation, we have $x^2 - 1 = 0$, from which we get $x = 1$ and $x = -1$. From the second equation, we have $y^2 - 9 = 0$, so that $y = 3$ and $y = -3$. We combine these solutions in all possible ways, and we have four critical points:

$$\big(1,3, f(1,3)\big), \qquad \big(1,-3, f(1,-3)\big), \qquad \big(-1,3, f(-1,3)\big), \quad \text{and} \quad \big(-1,-3, f(-1,-3)\big).$$

The z values are $f(1,3) = -49$, $f(1,-3) = 59$, $f(-1,3) = -45$ and $f(-1,-3) = 63$.

To classify these critical points, we use the second derivative test. The second derivatives are

$$f_{xx}(x, y) = 6x, \qquad f_{yy}(x, y) = 6y \quad \text{and} \quad f_{xy}(x, y) = 0.$$

Thus, using the formula, we have

$$D = f_{xx}(x, y) f_{yy}(x, y) - \left(f_{xy}(x, y)\right)^2 = (6x)(6y).$$

- When $x = 1$ and $y = 3$, we have $D = (6)(18)$, which is a positive number. Note that $f_{xx}(1,3)$ is also positive. Thus, the critical point $(1, 3, -49)$ is a minimum.

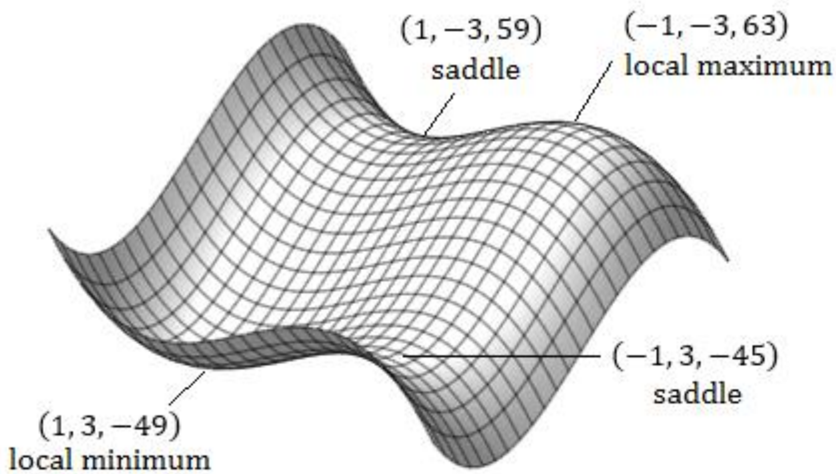
- When $x = 1$ and $y = -3$, we have $D = (6)(-18)$, which is a negative number. Thus, the critical point $(1, -3, 59)$ is a saddle point.

- When $x = -1$ and $y = 3$, we have $D = (-6)(18)$, which is a negative number. Thus, the critical point $(-1, 3, -45)$ is a saddle point.

- When $x = -1$ and $y = -3$, we have $D = (-6)(-18)$, which is a positive number. Note that $f_{xx}(-1, -3)$ is negative. Thus, the critical point $(-1, -3, 63)$ is a maximum.

Note that when finding *D*, it's not important to determine its actual value. It's more important to determine its sign. Thus, calculating (6)(18) is not as important as observing that the product of two positive values will be positive. Furthermore, by leaving the expression as (6)(18) rather than simplifying it, we can also quickly see that the value 6, representing $f_{xx}(x,y)$ when x = 1, is positive.

The graph of $z = f(x,y) = x^3 + y^3 - 3x - 27y + 7$ is below:



♦ • ♦ • ♦ • ♦ • ♦

Often, a system must be solved to determine critical points.

Example 29.5: Find the critical points of $z = g(x,y) = x^2 + y^2 - 4x + y + xy + 3$, and classify these points as minima, maxima or saddle.

Solution: We find the partial derivatives:

$$\begin{aligned} g_x &= 2x - 4 + y \\ g_y &= 2y + 1 + x. \end{aligned}$$

These are set equal to 0 and a linear system in two variables results:

$$\begin{aligned} 2x - 4 + y &= 0 \\ 2y + 1 + x &= 0 \end{aligned}, \quad \text{which simplifies as} \quad \begin{aligned} 2x + y &= 4 \\ x + 2y &= -1. \end{aligned}$$

We multiply the second equation by –2:

$$\begin{aligned} 2x + y &= 4 \\ -2x - 4y &= 2. \end{aligned}$$

Summing, we have $-3y = 6$, so that $y = -2$. Back substituting, we find that $x = 3$. Thus, $\big(3, -2, g(3, -2)\big)$ is the critical point. The z value is $g(3, -2) = -4$.

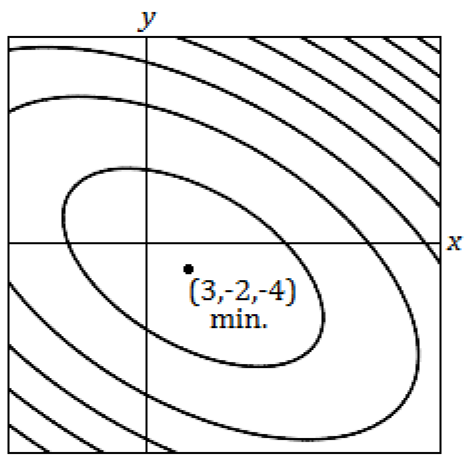
The second derivatives are

$$g_{xx}(x, y) = 2, \quad g_{yy}(x, y) = 2 \quad \text{and} \quad g_{xy}(x, y) = 1.$$

Using the second derivative test, we have

$$\begin{aligned} D &= g_{xx}(3, -2) g_{yy}(3, -2) - \big(g_{xy}(3, -2)\big)^2 \\ &= (2)(2) - 1^2 \\ &= 3. \end{aligned}$$

Since $D > 0$ and $g_{xx}(3, -2) > 0$, this point is a local minimum. The graph is a paraboloid with vertex $(3, -2, -4)$ opening upward. Thus, the point is also an absolute minimum.



♦ • ♦ • ♦ • ♦ • ♦

Example 29.6: Find the critical points of $z = f(x, y) = x^3 - y^3 - 2x^2 + xy + 3y$, and classify these points as minima, maxima or saddle.

Solution: The partial derivatives are

$$\begin{aligned} f_x &= 3x^2 - 4x + y \\ f_y &= -3y^2 + x + 3. \end{aligned}$$

Setting these to zero, we develop a non-linear system:

$$\begin{aligned} 3x^2 - 4x + y &= 0 \\ -3y^2 + x + 3 &= 0. \end{aligned}$$

Unlike the previous example, we cannot use the elimination method. Instead, we use substitution. In the first equation, solve for *y*:

$$y = 4x - 3x^2.$$

This is substituted into the second equation, then simplified:

$$\begin{aligned} -3(4x - 3x^2)^2 + x + 3 &= 0 \\ -3(16x^2 - 24x^3 + 9x^4) + x + 3 &= 0 \\ -27x^4 + 72x^3 - 48x^2 + x + 3 &= 0. \end{aligned}$$

Using a graphing calculator, we find four roots to this quartic equation. They are

$$x \approx -0.209, \quad x \approx 0.364, \quad x \approx 0.919 \quad \text{and} \quad x \approx 1.592.$$

Evaluating the equation $y = 4x - 3x^2$ at each of these *x* values, we have four critical points:

$$(-0.209, -0.967, -1.891), \quad (0.364, 1.059, 2.158),$$
$$(0.919, 1.142, 2.073) \quad \text{and} \quad (1.592, -1.235, -4.822).$$

The *z*-values were found by evaluating f at each *x* and *y* value. Note that the *z*-values alone do not provide enough information to classify these points as minimum, maximum or neither. We use the second derivative test.

The second derivatives are

$$f_{xx} = 6x - 4, \quad f_{yy} = -6y, \quad f_{xy} = 1.$$

Thus, we have

$$
\begin{aligned}
D &= f_{xx}(x_c, y_c) f_{yy}(x_c, y_c) - \left(f_{xy}(x_c, y_x)\right)^2 \\
&= (6x_c - 4)(-6y_c) - (1)^2,
\end{aligned}
$$

where (x_c, y_c) represent a critical point.

Each critical point is evaluated into the second derivative test formula:

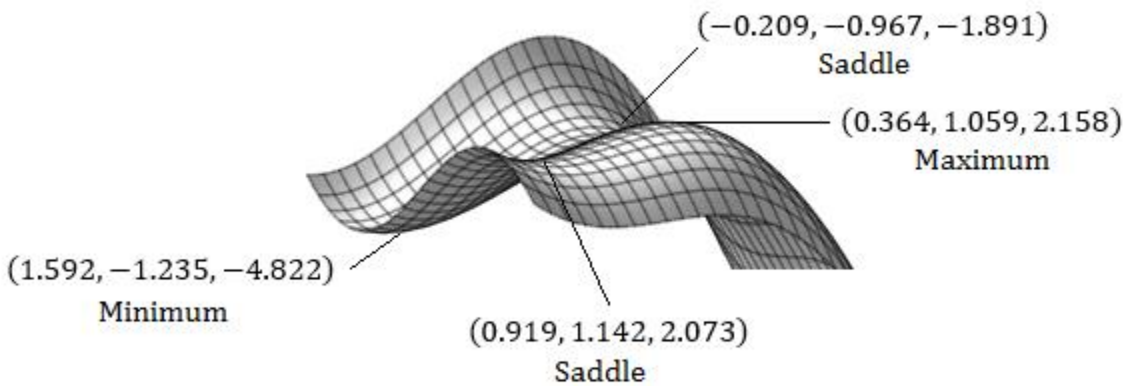
- At $(-0.209, -0.967, -1.891)$, we have $D = (6(-0.209) - 4)\big(-6(-0.967)\big) - 1 = -31.484$. Since D is negative, the point $(-0.209, -0.967, -1.891)$ is a saddle point.

- At $(0.364, 1.059, 2.158)$, we have $D = (6(0.364) - 4)\big(-6(1.059)\big) - 1 = 10.539$. Since D is positive and since f_{xx} is negative (as is f_{yy}), the point $(0.364, 1.059, 2.158)$ is a local maximum.

- At $(0.919, 1.142, 2.073)$, we have $D = (6(0.919) - 4)\big(-6(1.142)\big) - 1 = -11.373$. Since D is negative, the point $(0.919, 1.142, 2.073)$ is a saddle point.

- At $(1.592, -1.235, -4.822)$, we have $D = (6(1.592) - 4)\big(-6(-1.235)\big) - 1 = 40.14$. Since D is positive and since f_{xx} is positive (as is f_{yy}), the point $(1.592, -1.235, -4.822)$ is a local minimum.

Again, note that the z-values play no role in classifying these points.



♦ • ♦ • ♦ • ♦ • ♦