38. Triple Integration over Rectangular Regions

A rectangular solid region S in R^3 can be defined by three compound inequalities,

$$a_1 \leq x \leq a_2, \quad b_1 \leq y \leq b_2, \quad c_1 \leq z \leq c_2,$$

where a_1, a_2, b_1, b_2, c_1 and c_2 are constants. A function of three variables $w = f(x, y, z)$ that is continuous over S can be integrated as a **triple integral**:

$$\iiint_S f(x,y,z)\, dV = \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} f(x,y,z)\ dz\, dy\, dx.$$

Observe that the integrals are nested: the inside integral, labeled dz, is associated with the bounds $c_1 \leq z \leq c_2$, and similarly as one works outward.

The volume element is labeled dV and there are six possible orderings of the differentials dx, dy and dz, whose product is equivalent to dV:

$$\begin{array}{lll} dz\, dy\, dx, & dz\, dx\, dy, & dy\, dz\, dx, \\ dy\, dx\, dz, & dx\, dz\, dy, & dz\, dy\, dz. \end{array}$$

When all bounds are constant, no particular ordering is more advantageous than any other. All six possible orderings will give the same result.

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Example 38.1: Evaluate

$$\int_{-1}^{2} \int_{1}^{3} \int_{-2}^{4} (x + 2yz^2)\, dz\, dy\, dx.$$

Solution: The inner-most integral is evaluated. Since the integrand is being antidifferentiated with respect to *z*, the variables *x* and *y* are treated as constants or coefficients for the moment:

$$\begin{aligned} \int_{-2}^{4} (x + 2yz^2)\, dz &= \left[xz + \frac{2}{3}yz^3\right]_{-2}^{4} \\ &= \left(x(4) + \frac{2}{3}y(4)^3\right) - \left(x(-2) + \frac{2}{3}y(-2)^3\right) \\ &= \left(4x + \frac{128}{3}y\right) - \left(-2x - \frac{16}{3}y\right) \\ &= 6x + 48y. \end{aligned}$$

This is now integrated with respect to *y* (the "middle" integral). The *x* is still treated as a constant or coefficient in this step:

$$\int_1^3 (6x + 48y)\, dy = [6xy + 24y^2]_1^3$$
$$= (6x(3) + 24(3)^2) - (6x(1) + 24(1)^2)$$
$$= 12x + 192.$$

Lastly, this is integrated with respect to *x*, the "outer" integral:

$$\int_{-1}^2 (12x + 192)\, dx = [6x^2 + 192x]_{-1}^2$$
$$= \left(6(2)^2 + 192(2)\right) - \left(6(-1)^2 + 192(-1)\right)$$
$$= 594.$$

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How do we interpret answers obtained from a triple integral? Analogous to a single-variable integral (the definite integral is the *area* between a curve and over an interval on the input axis) and a two-variable double integral (the definite double integral is the *volume* between a surface and a region in the input plane), a three-variable continuous function $w = f(x, y, z)$ evaluated over a triple integral gives a "volume" between the graph of f and the region *S* in R^3 over which it is being integrated. However, the graph of $w = f(x, y, z)$ is actually embedded within R^4, so it is not easy to visualize this four-dimensional analog to area or volume. Nevertheless, it is a reasonable interpretation.

One immediate corollary is to allow the integrand to be 1. In such a case, we get a volume integral, where $\iiint_S\ 1\, dV$ is the volume of *S*.

Example 38.2: Evaluate

$$\int_{-3}^5 \int_2^4 \int_{-1}^8 1\, dz\, dy\, dx.$$

Solution: Working inside out, we have $\int_{-1}^8 1\, dz = [z]_{-1}^8 = 8 - (-1) = 9$. Then, we have $9\int_2^4 dy = 9[y]_2^4 = 9(4 - 2) = 18$. Lastly, we have $18\int_{-3}^5 dx = 18[x]_{-3}^5 = 18\left(5 - (-3)\right) = 144$.

This is the volume of the rectangular solid region in R^3 in which length *x* is 8 units, length *y* is 2 units, and length *z* is 9 units. Not surprisingly, $(8)(2)(9) = 144$ cubic units.

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Example 38.3: Evaluate

$$\int_2^5 \int_0^4 \int_{-1}^3 x^2yz^3 \; dx\, dy\, dz.$$

Solution: Note the order of integration. The inside integral is integrated with respect to *x*. The yz^3 factors are treated as a constant and moved outside the integral:

$$\begin{aligned}\int_{-1}^3 x^2yz^3 \; dx &= yz^3 \int_{-1}^3 x^2 \; dx \\ &= yz^3 \left[\frac{1}{3}x^3\right]_{-1}^3 \\ &= yz^3\left(\left(\frac{1}{3}(3)^3\right) - \left(\frac{1}{3}(-1)^3\right)\right) = \frac{28}{3}yz^3.\end{aligned}$$

This expression is now integrated with respect to *y*, the middle integral. We can move the $\frac{28}{3}z^3$ to the front of the integral:

$$\begin{aligned}\int_0^4 \left(\frac{28}{3}yz^3\right) \; dy &= \frac{28}{3}z^3 \int_0^4 y\; dy \\ &= \frac{28}{3}z^3\left[\frac{1}{2}y^2\right]_0^4 \\ &= \frac{28}{3}z^3(8) \\ &= \frac{224}{3}z^3.\end{aligned}$$

Lastly, this expression is integrated with respect to *z*:

$$\begin{aligned}\int_2^5 \left(\frac{224}{3}z^3\right) \; dz &= \frac{224}{3}\int_2^5 z^3 \; dz \\ &= \frac{224}{3}\left[\frac{1}{4}\; z^4\right]_2^5 \\ &= \frac{224}{3}\left(\left(\frac{1}{4}(5)^4\right) - \left(\frac{1}{4}(2)^4\right)\right) \\ &= \frac{224}{3}\left(\frac{609}{4}\right) \\ &= \frac{136{,}416}{12} = 11{,}368.\end{aligned}$$

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If the integrand is held by multiplication so that it can be written as $f(x,y,z) = g(x)h(y)k(z)$, and the bounds are constants, then

$$\int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} f(x,y,z)\ dz\,dy\,dx = \int_{a_1}^{a_2}\int_{b_1}^{b_2}\int_{c_1}^{c_2} g(x)h(y)k(z)\ dz\,dy\,dx$$
$$= \left(\int_{a_1}^{a_2} g(x)\,dx\right)\left(\int_{b_1}^{b_2} h(y)\,dy\right)\left(\int_{c_1}^{c_2} k(z)\,dz\right).$$

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Example 38.4: Use the shortcut shown above to evaluate

$$\int_2^5\int_0^4\int_{-1}^3 x^2yz^3\ dx\,dy\,dz.$$

Solution: Since the bounds are constants and the integrand is held by multiplication, the above triple integral can be rewritten as a product of three single-variable integrals, and evaluated individually:

$$\left(\int_{-1}^3 x^2\,dx\right)\left(\int_0^4 y\,dy\right)\left(\int_2^5 z^3\,dz\right) = \left(\left[\frac{1}{3}x^3\right]_{-1}^3\right)\left(\left[\frac{1}{2}y^2\right]_0^4\right)\left(\left[\frac{1}{4}z^4\right]_2^5\right)$$
$$= \left(\frac{1}{3}(3^3-(-1)^3)\right)\left(\frac{1}{2}(4^2-0^2)\right)\left(\frac{1}{4}(5^4-2^4)\right)$$
$$= \left(\frac{28}{3}\right)(8)\left(\frac{609}{4}\right) = 11{,}368.$$

Note that this shortcut would not work with the first example, $\int_{-1}^2\int_1^3\int_{-2}^4(x+2yz^2)\ dz\ dy\ dx$.

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39. Triple Integration over Non-Rectangular Regions of Type I

A solid region S in R^3 is considered to be **Type I** if there is no ambiguity as to any of its bounds of integration in such a way that one triple integral is sufficient to describe S. Because there are six possible orderings of the variables of integration, it is possible that one ordering may result in a non-Type I region, while another ordering may result in a Type I region. ever possible, choose a Type I ordering of integration.

For example, all rectangular solid regions in the previous examples are Type I, in any ordering of the differentials.

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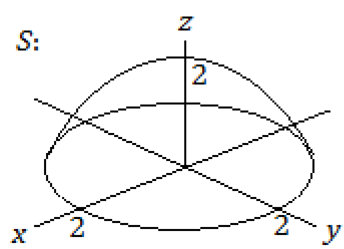
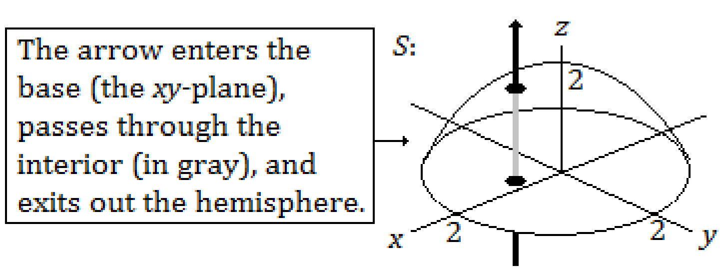
Example 39.1: Find $\iiint_S dV$, where S is a solid hemisphere, centered at the origin, of radius 2 such that $z \geq 0$.

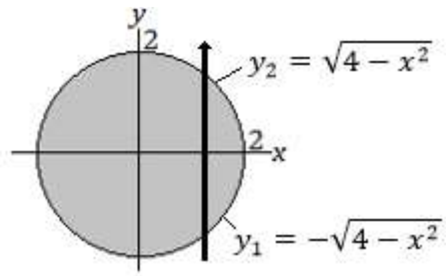
Solution: Sketch the solid. The restriction $z \geq 0$ means all points are on or above the *xy*-plane:



Now, select an ordering of integration. Let's try $dz\, dy\, dx$, so that the first integral is evaluated with respect to *z*. Sketch an arrow in the positive *z* direction so that it enters the solid through one surface, and exits through another. It is important to observe that in this case, there is *no* ambiguity as to where such an arrow would enter or exit the solid: it *must* enter through the surface $z_1 = 0$ (the *xy*-plane) and *must* exit through $z_2 = \sqrt{4 - x^2 - y^2}$, the hemisphere. These will be the bounds for the dz integral.



Now, we concentrate on the region defined by the *x* and *y* variables. This is the "footprint" of the solid on the *xy*-plane, and is a disk of radius 2, centered at the origin:



If we next choose to integrate with respect to *y*, we draw an arrow in the positive *y* direction. It will enter the region through the lower half of the circle, $y_1 = -\sqrt{4 - x^2}$, and exit through the upper half, $y_2 = \sqrt{4 - x^2}$. There is no ambiguity as to where this arrow enters or exits the region. It is of Type I as well.

Lastly, the bounds for *x* are constants: $-2 \leq x \leq 2$. The triple integral is

$$\iiint_S dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx.$$

This is a volume integral (since the integrand is 1), representing the volume of the hemisphere of radius 2. Using geometry, the volume of a hemisphere is $\frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$. Thus, when $r = 2$, we have $\frac{1}{2}\left(\frac{4}{3}\pi r^3\right) = \frac{2}{3}\pi(2)^3 = \frac{16}{3}\pi$:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx = \frac{16}{3}\pi.$$

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The Legal Form of a Triple Integral

Triple integrals follow the form shown below:

$$\int_a^b \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

Note the ordering of integration: *z* first, then *y*, then *x*. If this ordering is chosen, then the innermost integral will have bounds that may contain *x* and *y*, possibly both:

$$z_1(x,y) \leq z \leq z_2(x,y).$$

The next integral, with respect to *y*, may have bounds that contain *x*, but not *z*:

$$y_1(x) \leq y \leq y_2(x).$$

The last (outermost) integral with respect to *x*, has bounds that are constants:

$$a \leq x \leq b.$$

The ordering of integration "drives" the bounds, so to speak. The following is a legal triple integral but in a different ordering of integration:

$$\int_{-1}^4 \int_{-x}^{3x} \int_0^{x+z} (x^2 + z)\, dy\, dz\, dx.$$

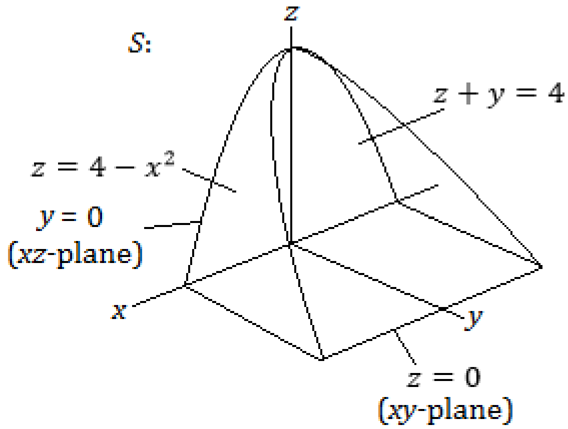
Note that the innermost integral with respect to *y* has bounds that may contain *x* or *z* (or both), while the middle integral, with respect to *z*, has bounds that may contain *x*, but not *y*. The outer integral's bounds must be constant.

This is an "illegal" triple integral:

$$\int_0^2 \int_{x+z}^{2y} \int_{-y^2}^{x} (\sin(xyz) + x)\, dz\, dy\, dx.$$

The innermost integral is legal: the bounds with respect to *z* may contain *x* or *y* (or both). However, the middle integral, with respect to *y*, cannot contain itself as a variable, nor *z*, since *z* is "done" by the time we evaluate this middle integral.

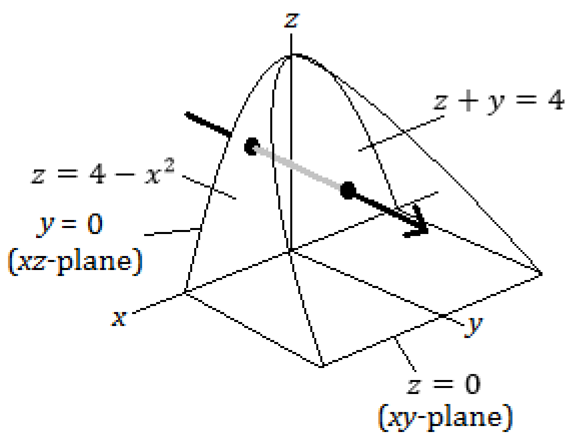
Example 39.2: Solid S is shown below. Let $f(x, y, z)$ be a generic integrand.



a) Set up a triple integral over S in the *dy dz dx* ordering.
b) Set up a triple integral over S in the *dx dy dz* ordering.
c) Explain why any ordering starting with *dz* is not of Type I.

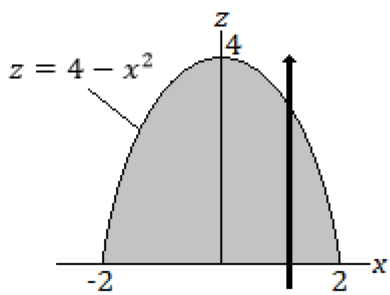
Solution:

a) Sketch an arrow in the positive y direction:



This arrow enters the solid at the xz-plane ($y_1 = 0$), passes through the interior (gray), and exits out the plane $z + y = 4$, or $y_2 = 4 - z$. These are the bounds for y.

Next, we look at the footprint of the solid as projected onto the *xz*-plane. Variable *y* is no longer needed.

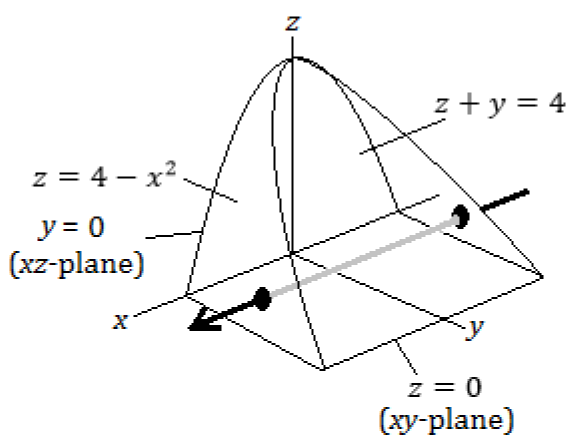


This region is Type I. The *z*-bounds, as shown by the arrow above, are $0 \leq z \leq 4 - x^2$, and the *x* bounds are constants, $-2 \leq x \leq 2$. Thus, the triple integral is

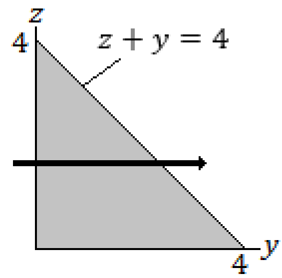
$$\int_{-2}^{2} \int_{0}^{4-x^2} \int_{0}^{4-z} f(x, y, z)\, dy\, dz\, dx.$$

Note that this integral is "legal". Do you agree?

b) For the *dx dy dz* ordering, draw an arrow in the positive *x* direction. It enters the region through the parabolic sheet $x_1 = -\sqrt{4 - z}$ and exits through $x_2 = \sqrt{4 - z}$.



Variable *x* is "done". We now look at the footprint of the solid projected onto the *yz* plane, and since the middle integral will be with respect to *y*, we sketch an arrow in the positive *y* direction.

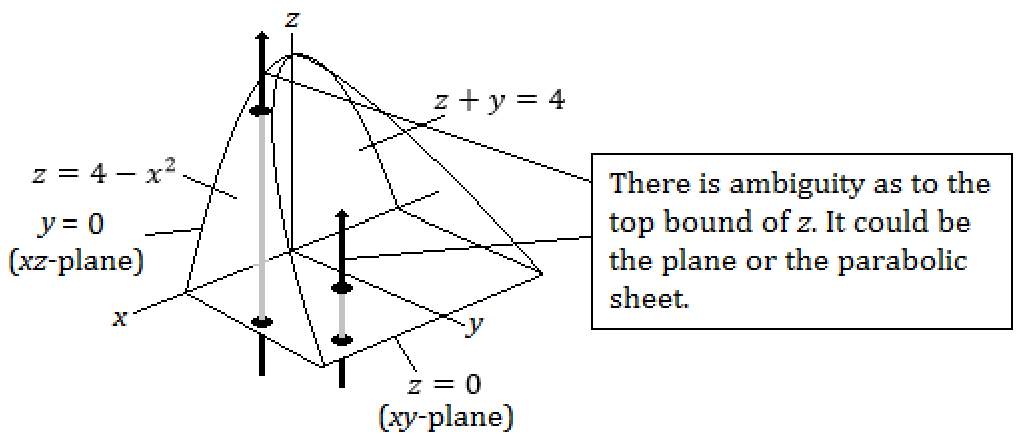


This region is also Type I. An arrow drawn in the positive y direction enters it at $y_1 = 0$ (the z axis) and exits through the line $y_2 = 4 - z$. Finally, the bounds on z are $0 \le z \le 4$. The triple integral is

$$\int_0^4 \int_0^{4-z} \int_{-\sqrt{4-z}}^{\sqrt{4-z}} f(x,y,z)\, dx\, dy\, dz.$$

Study this integral to convince yourself it is legal.

c) Any ordering starting with *dz* is not of Type I because an arrow drawn in the positive z direction may exit through the plane $z = 4 - y$, or the parabolic sheet $z = 4 - x^2$. Because there is ambiguity as to z's bounds, this solid is not of Type I if starting the integration with respect to z. In such a case, it's wise to find a different ordering.



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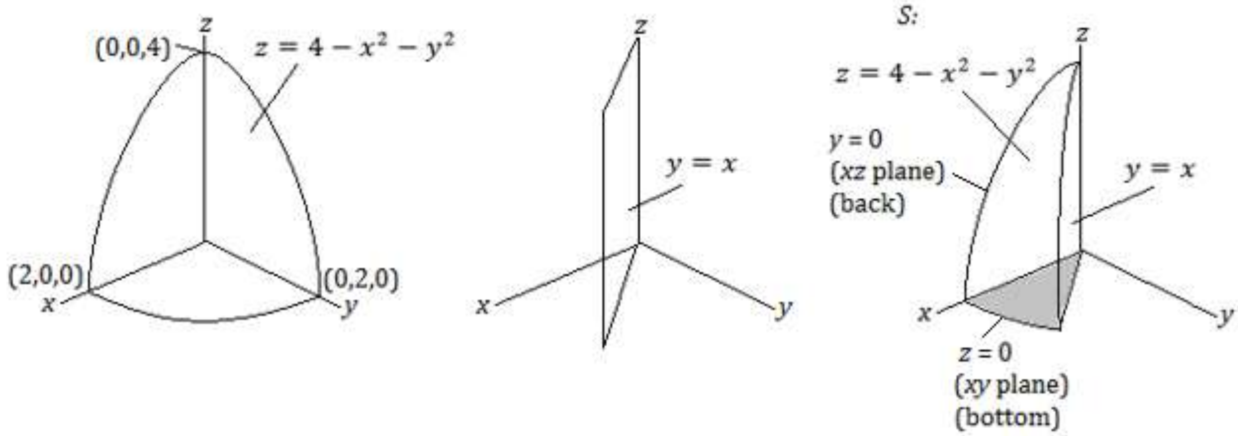
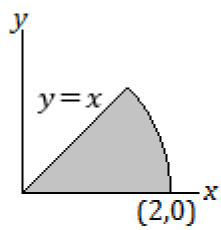
Example 39.3: Solid S is bounded by the surface $z = 4 - x^2 - y^2$, the plane $y = x$, the *xy*-plane and the *xz*-plane in the first octant. Find this solid's volume.

Solution: It is important to visualize the solid. The surface $z = 4 - x^2 - y^2$ is a paraboloid with vertex (0,0,4) that opens downward (left image below). The plane $y = x$ can be seen as the line $y = x$ in R^2, then extended into the *z*-direction (middle image, below).



If we choose to integrate with respect to *z* first, there will be no ambiguity in the bounds. The bounds for *z* will be $0 \leq z \leq 4 - x^2 - y^2$. The footprint of this region on the *xy*-plane is a circular wedge:



We use polar coordinates to describe this region. Recalling that $x = r\cos\theta$ and $y = r\sin\theta$, then this region's bounds are $0 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{4}$. However, since we have replaced variables *x* and *y* with r and θ, the top bound for *z*, which is $4 - x^2 - y^2$, is rewritten as $4 - (x^2 + y^2) = 4 - r^2$.

Thus, the volume is given by the triple integral below, with 1 as the integrand. Note the Jacobian r is also present in the integral.

$$\int_0^{\pi/4}\int_0^2\int_0^{4-r^2} 1\ dz\, r\, dr\, d\theta.$$

The inside integral is evaluated first:

$$\int_0^{4-r^2} 1 \; dz = 4 - r^2.$$

This is then integrated with respect to r:

$$\int_0^2 (4-r^2) r \, dr = \int_0^2 (4r - r^3) \, dr = \left[2r^2 - \frac{1}{4}r^4\right]_0^2 = 8 - 4 = 4.$$

Lastly, the outside integral is evaluated:

$$\int_0^{\pi/4} 4 \, d\theta = 4\left(\frac{\pi}{4}\right) = \pi.$$

The solid has a volume of π cubic units.

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The previous example, in which the variables *x* and *y* were replaced with r and θ, is an example of integrating in **cylindrical coordinates**. Note that the variable *z* was left unchanged, but its bounds, which included variables *x* and *y*, had to be adjusted to include the new variables r and θ. In general, such a triple integral in cylindrical coordinates is given by

$$\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{z_1(r,\theta)}^{z_2(r,\theta)} f(r,\theta,z) \, dz \, r \, dr \, d\theta.$$

Typically, the inside integral, with respect to *z*, is integrated first.

This does not exclude situations where two of the other variables may be exchanged for r and θ. For example, if variables y and z are defined over a region that is better described using polar coordinates, then *x* is left alone, but the bounds for *x* are adjusted to include r and θ, and a triple integral in cylindrical coordinates would be given by

$$\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{x_1(r,\theta)}^{x_2(r,\theta)} f(x,r,\theta) \, dx \, r \, dr \, d\theta.$$

Furthermore, the transformation is arbitrary: we can declare that $y = r\cos\theta$ and $z = r\sin\theta$, or that $y = r\sin\theta$ and $z = r\cos\theta$. As long as the bounds are handled correctly, either transformation is acceptable.

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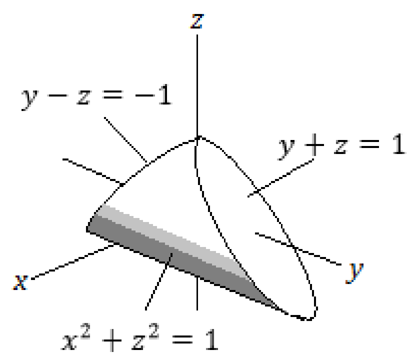
Example 39.4: A cylinder, $x^2 + z^2 = 1$, is intersected by the planes $y + z = 1$ and $y - z = -1$. Find the volume of this intersecting region.

Solution: Below is a sketch of the region. Note that the cylinder $x^2 + z^2 = 1$ can be viewed as a circle of radius 1, centered at the origin, on the *xz*-plane, then extended into the positive and negative *y* directions. The planes $y + z = 1$ and $y - z = -1$ can be viewed as lines on the *yz*-plane, then extended into the positive and negative *x* directions.



Visualizing an arrow in the positive *y* direction, it enters the solid through the plane $y - z = -1$, or $y_1 = z - 1$, then exits the solid through the plane $y + z = 1$, or $y_2 = 1 - z$. Note that variables *x* and *z* form a circular region on the *xz*-plane, and this suggests we may want to exchange them for r and θ, and integrate with respect to *y* first. The bounds for r are $0 \le r \le 1$ and the bounds for θ are $0 \le \theta \le 2\pi$. An initial triple integral in cylindrical coordinates is given by

$$\int_0^{2\pi} \int_0^1 \int_{z-1}^{1-z} (1)\, dy\, r\, dr\, d\theta\,.$$

However, this is not quite correct. The bounds for *y* need to be written in terms of r and θ. If we define $x = r\cos\theta$ and $z = r\sin\theta$, the triple integral is now properly written as

$$\int_0^{2\pi} \int_0^1 \int_{r\sin\theta-1}^{1-r\sin\theta} (1)\, dy\, r\, dr\, d\theta\,.$$

The inside integral is evaluated first:

$$\int_{r\sin\theta-1}^{1-r\sin\theta} (1)\, dy = [y]_{r\sin\theta-1}^{1-r\sin\theta} = (1 - r\sin\theta) - (r\sin\theta - 1) = 2 - 2r\sin\theta.$$

This is integrated with respect to r:

$$\int_0^1 (2 - 2r\sin\theta)\; dr = [2r - r^2\sin\theta]_0^1 = 2 - \sin\theta.$$

Finally, this is integrated with respect to θ:

$$\int_0^{2\pi} (2 - \sin\theta)\, d\theta = [2\theta + \cos\theta]_0^{2\pi}$$

$$= (2(2\pi) + \cos(2\pi)) - (2(0) + \cos(0)) \quad \begin{cases}\text{Recall that } \cos(2\pi) = 1 \\ \text{and } \cos(0) = 1.\end{cases}$$

$$= 4\pi + 1 - 1 = 4\pi.$$

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Finding Volumes using Double Integrals and Triple Integrals. What's the Difference?

Suppose we want to determine the volume contained between the surface (graph) of $z = f(x, y)$ and the plane $z = 0$ (the *xy*-plane), where the region of integration in the *xy*-plane is defined by $y_1(x) \le y \le y_2(x)$ and $a \le x \le b$. Using a double integral, we would write

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x, y)\, dy\, dx.$$

Using a triple integral, we would write

$$\int_a^b \int_{y_1(x)}^{y_2(x)} \int_0^{f(x,y)} dz\, dy\, dx.$$

Observe that the innermost integral is $\int_0^{f(x,y)} dz = [z]_0^{f(x,y)} = f(x, y)$.

This is a common tactic, in which the integrand can be rewritten as the bound(s) of an entirely new integral. For example, if we wanted to find the volume between the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$, we could represent this volume by a double integral:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left((8 - x^2 - y^2) - (x^2 + y^2)\right) dy\, dx,$$

where the region of integration in the *xy*-plane is a circle of radius 2, and the integrand is written as "top surface" minus "bottom surface". As a triple integral, we have

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{8-x^2-y^2} dz\, dy\, dx.$$

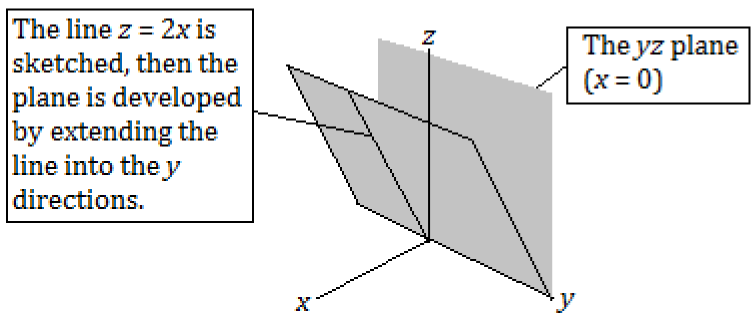
Any variable expression can be rewritten in integral form. For example, $x^2 = \int_0^{x^2} dt$. We can be creative too. For example, $2x^3 - xy = \int_0^{2x^3-xy} dt$ or $\int_{xy}^{2x^3} dt$.

Example 39.5: Consider the triple integral

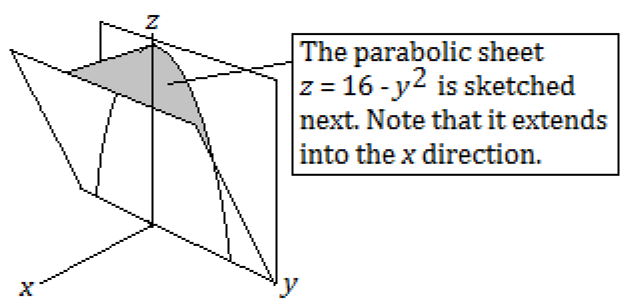
$$\int_{-4}^{4}\int_{0}^{16-y^2}\int_{0}^{\frac{1}{2}z} f(x,y,z)\,dx\,dz\,dy.$$

Rewrite this integral in the $dy\,dz\,dx$ ordering.

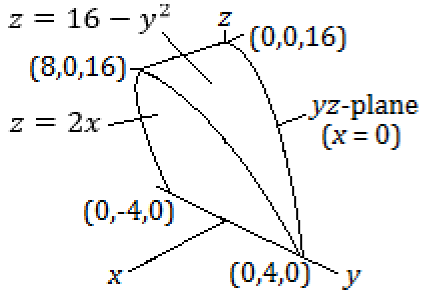
Solution: From the bounds, we can develop the solid S over which the integral is defined. Working inside out, we see that the bounds for x are $0 \le x \le \frac{1}{2}z$. This suggests that one bounding surface is the yz-plane, since $x = 0$. The other bounding surface is the plane, $x = \frac{1}{2}z$. It is important to remember that the bounding surfaces exist in R^3. Note that $x = \frac{1}{2}z$ is the same as $z = 2x$.



Now, the middle integral suggests that the bounds for z are the xy-plane ($z = 0$) and the parabolic sheet, $z = 16 - y^2$:

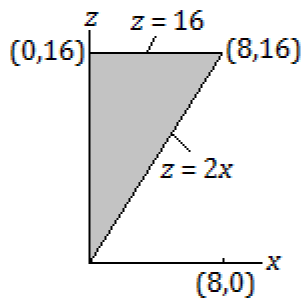


From this, the shape of the solid can be inferred. Strategic points are identified.



To rewrite the integral in the *dy dz dx* ordering, visualize an arrow in the positive *y* direction. There is no ambiguity where it enters or exits the solid. It enters through one half of the parabolic sheet $y_1 = -\sqrt{16-z}$ and exits through the other half, $y_2 = \sqrt{16-z}$. These are the bounds for *y*.

Now, we view the footprint of the solid as it appears projected onto the *xz*-plane. It will appear as a triangle, as shown below:



Integrating next with respect to *z*, the lower bound is $z_1 = 2x$ and the upper bound is $z_2 = 16$. Lastly, the bounds for *x* are $0 \le x \le 8$. Thus, the triple integral

$$\int_{-4}^{4}\int_{0}^{16-y^2}\int_{0}^{\frac{1}{2}z} f(x,y,z)\,dx\ dz\ dy$$

is equivalent to

$$\int_{0}^{8}\int_{2x}^{16}\int_{-\sqrt{16-z}}^{\sqrt{16-z}} f(x,y,z)\,dy\,dz\,dx.$$

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Example 39.6: Let solid *S* be a tetrahedron in the first octant with vertices (0,0,0), (2,0,0), (0,4,0) and (0,0,8). Set up all six possible triple integrals $\iiint_S f(x,y,z)\,dV$.

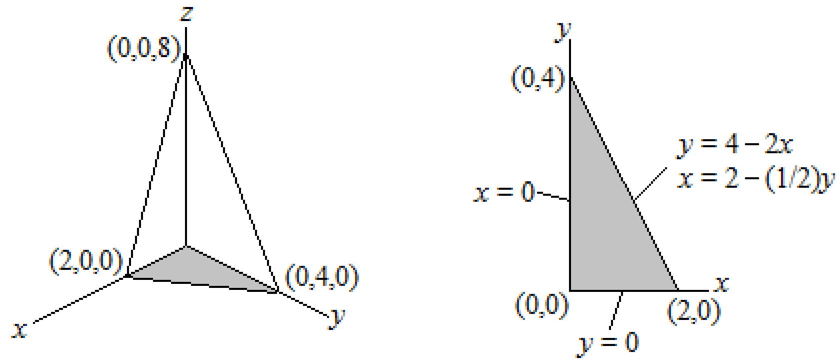
Solution: The equation of the plane that passes through the points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ is given by

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \qquad \text{(See Example 13.6)}$$

Thus, the equation of the plane passing through (2,0,0), (0,4,0) and (0,0,8) is

$$\frac{x}{2}+\frac{y}{4}+\frac{z}{8}=1.$$

If the inside integral is evaluated with respect to *z*, then we solve for *z*, getting $z = 8 - 4x - 2y$. The bounds of this integral are $0 \le z \le 8 - 4x - 2y$. This leaves a triangular region in the *xy*-plane with vertices (0,0), (2,0) and (0,4), shown below.



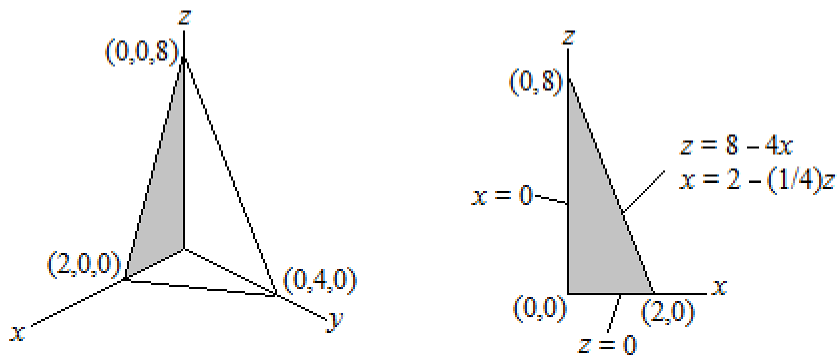
Integrating next with respect to *y*, the bounds are $0 \le y \le 4 - 2x$, where $0 \le x \le 2$. The triple integral is

$$\int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y} f(x, y, z)\, dz\, dy\, dx.$$

If we integrate next with respect to *x*, its bounds are $0 \le x \le 2 - \frac{1}{2}y$, then $0 \le y \le 4$, and the triple integral is

$$\int_0^4 \int_0^{2-(1/2)y} \int_0^{8-4x-2y} f(x, y, z)\, dz\, dx\, dy.$$

Repeating this process from the start, suppose now that the inside integral is evaluated with respect to *y*. Solve for *y*, getting $y = 4 - 2x - \frac{1}{2}z$. The bounds of this integral are $0 \le y \le 4 - 2x - \frac{1}{2}z$. This leaves a triangular region in the *xz*-plane with vertices (0,0), (2,0) and (0,8), shown below.



Integrating next with respect to z, the bounds are $0 \leq z \leq 8 - 4x$, where $0 \leq x \leq 2$. The triple integral is

$$\int_0^2 \int_0^{8-4x} \int_0^{4-2x-(1/2)z} f(x,y,z)\, dy\, dz\, dx.$$

Integrating next with respect to x, the bounds are $0 \leq x \leq 2 - \frac{1}{4}z$, where $0 \leq x \leq 8$. The triple integral is

$$\int_0^8 \int_0^{2-(1/4)z} \int_0^{4-2x-(1/2)z} f(x,y,z)\, dy\, dx\, dz.$$

You should verify that the triple integral in the $dx\, dy\, dz$ ordering is

$$\int_0^8 \int_0^{4-(1/2)z} \int_0^{2-(1/2)y-(1/4)z} f(x,y,z)\, dx\, dy\, dz,$$

and that the triple integral in the $dx\, dz\, dy$ ordering is

$$\int_0^4 \int_0^{8-2y} \int_0^{2-(1/2)y-(1/4)z} f(x,y,z)\, dx\, dz\, dy.$$

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