27. Tangent Planes & Approximations

If $z = f(x, y)$ is a differentiable surface in R^3 and (x_0, y_0, z_0) is a point on this surface, then it is possible to construct a plane passing through this point, tangent to the surface of f.

Recall that a plane is constructed by determining a vector $\mathbf{n} = \langle a, b, c \rangle$ normal to the plane and identifying a point (x_0, y_0, z_0) on the plane. With this information, the equation of the plane is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

To find this desired normal vector **n**, we temporarily write $z = f(x, y)$ as a function of three variables, $F(x, y, z) = f(x, y) - z$, noting that the equation $f(x, y) - z = 0$ is now a level curve of the graph of F. Recall from Section 26 that the vectors in the gradient of F will be orthogonal to all level curves of the graph of F. We start with $\nabla F = \langle F_x, F_y, F_z \rangle$, and observe that $F_x = f_x$ and that $F_y = f_y$, and since we wrote $F(x, y, z) = f(x, y) - z$, that $F_z = -1$. Therefore,

$$\mathbf{n} = \langle a, b, c \rangle = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle.$$

Tying this all together, the **equation of the tangent plane** to a point (x_0, y_0, z_0) on the surface of $z = f(x, y)$ is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

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Example 27.1: Let $z = f(x, y) = x^2 + 2xy^3$. Find the equation of the tangent plane to f when $x_0 = 1$ and $y_0 = 2$.

Solution: When $x_0 = 1$ and $y_0 = 2$, then $z_0 = f(x_0, y_0) = f(1,2) = (1)^2 + 2(1)(2)^3 = 17$. Thus, the point of tangency is $(x_0, y_0, z_0) = (1,2,17)$.

The partial derivatives are $f_x(x, y) = 2x + 2y^3$ and $f_y(x, y) = 6xy^2$. Evaluated at $x_0 = 1$ and $y_0 = 2$, we have $f_x(1,2) = 18$ and $f_y(1,2) = 24$. Thus, the plane of tangency is

$$18(x - 1) + 24(y - 2) - (z - 17) = 0.$$

Simplified, the plane is $18x + 24y - z = 49$, or with z isolated, we obtain $z = 18x + 24y - 49$.

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Example 27.2: Find the equation of the tangent plane to $z = g(x, y) = \frac{2x+y}{3y^2}$ when $x_0 = -2$ and $y_0 = 3$.

Solution: The point of tangency is $(x_0, y_0, z_0) = \left(-2,3, -\frac{1}{27}\right)$, where $z_0 = \frac{2(-2)+(3)}{3(3)^2} = -\frac{1}{27}$.

The partial derivatives are

$$g_x(x, y) = \frac{2}{3y^2} \quad \text{and} \quad g_y(x, y) = -\frac{y + 4x}{3y^3}.$$

Evaluated at $x_0 = -2$ and $y_0 = 3$, we have $g_x(-2,3) = \frac{2}{27}$ and $g_y(-2,3) = \frac{5}{81}$. Thus, the equation of the plane of tangency is

$$\frac{2}{27}\left(x - (-2)\right) + \frac{5}{81}(y - 3) - \left(z - \left(-\frac{1}{27}\right)\right) = 0.$$

Multiplying by 81 to clear fractions and then distributing to clear parentheses, the equation is simplified to $6x + 5y - 81z = 6$.

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This process can be extended to surfaces in higher dimension.

Example 27.3: Find the equation of the tangent plane to $w = f(x, y, z) = x^2y^3z^4$ at $(2,1, -2,64)$.

Solution: The partial derivatives are evaluated at $x_0 = 2$, $y_0 = 1$ and $z_0 = -2$:

$$f_x(x, y, z) = 2xy^3z^4 \to f_x(2,1, -2) = 64,$$
$$f_y(x, y, z) = 3x^2y^2z^4 \to f_y(2,1, -2) = 192,$$
$$f_z(x, y, z) = 4x^2y^3z^3 \to f_z(2,1, -2) = -128.$$

Thus, the plane of tangency is

$$64(x - 2) + 192(y - 1) - 128\left(z - (-2)\right) - 1(w - 64) = 0.$$

Solving for *w*, we have

$$w = 64(x - 2) + 192(y - 1) - 128(z + 2) + 64.$$

Simplified, we have $w = 64x + 192y - 128z - 512$.

Tangent planes can be used to estimate values on the surface of a multi-variable function f.

Example 27.4: Given that $z = 18x + 24y - 49$ is the equation of the plane tangent to the surface $f(x, y) = x^2 + 2xy^3$ when $x_0 = 1$ and $y_0 = 2$, estimate the value of $f(1.1, 1.9)$.

Solution: Since planes consist only of linear and constant terms, it is usually easier to evaluate points on a plane rather than points on a surface. In this case, we have

$$z = 18(1.1) + 24(1.9) - 49 = 16.4.$$

Observe that the point $(1.1, 1.9, 16.4)$ lies on the tangent plane, not on the surface of f. However, if we were to evaluate f at $x = 1.1$ and $y = 1.9$, we obtain

$$f(1.1, 1.9) = (1.1)^2 + 2(1.1)(1.9)^3 = 16.2998.$$

The estimated value of 16.4 is an excellent approximation of the actual value of 16.2998. Using planes to estimate values on a surface requires that the point of evaluation be "close" to the point of tangency. In this example, 1.1 is close to 1, and 1.9 is close to 2. However, suppose that we wanted to use the tangent plane to estimate $f(1.5, 2.4)$. We get

$$z = 18(1.5) + 24(2.4) - 49 = 35.6,$$

The actual point on the surface is $f(1.5, 2.4) = 43.722$. We see that the estimated value of *z* is not close to the actual value of *z*.

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Example 27.5: Given the surface $w = f(x, y, z) = x^2y^3z^4$ at $(2, 1, -2, 64)$ in Example 27.3, estimate the value of $w = f(2.01, 0.99, -1.98)$.

Solution: We use the equation $w = 64(x - 2) + 192(y - 1) - 128(z + 2) + 64$ from Example 27.3. We then substitute *x* = 2.01, *y* = 0.99 and *z* = –1.98:

$$\begin{aligned} w &= 64(2.01 - 2) + 192(0.99 - 1) - 128(-1.98 + 2) + 64 \\ &= 64(0.01) + 192(-0.01) - 128(0.02) + 64 \\ &= 0.64 - 1.92 - 2.56 + 64 \\ &= -3.84 + 64 \\ &= 60.16. \end{aligned}$$

The actual *w*-value is $w = f(2.01, 0.99, -1.98) = (2.01)^2(0.99)^3(-1.98)^4 = 60.25$. The estimation is very close to the actual value.

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Example 27.6: Find the acute angle that the tangent plane of $f(x, y) = 3x^2 - 2y$, when $x_0 = -2$ and $y_0 = 3$, makes with the *xy*-plane.

Solution: The partial derivatives are $f_x(x, y) = 6x$ and $f_y(x, y) = -2$. Thus, the normal vector **n** is

$$\begin{aligned}\mathbf{n} &= \langle f_x(-2,3), f_y(-2,3), -1\rangle \\ &= \langle 6(-2), -2, -1\rangle \\ &= \langle -12, -2, -1\rangle.\end{aligned}$$

The *xy*-plane has two "convenient" normal vectors, the positive *z*-axis represented by the vector $\mathbf{z}^+ = \langle 0,0,1\rangle$ and the negative *z*-axis represented by the vector $\mathbf{z}^- = \langle 0,0,-1\rangle$. Since **n** points in the direction of the negative *z*-axis, we will compare **n** to $\mathbf{z}^-$.

Recall that the angle between two planes is the same as the angle between its normal vectors, and that two planes always meet acutely (except when they are orthogonal). Thus, to find the angle between the *xy*-plane and the plane of tangency, it is sufficient to determine the angle between the two normal vectors.

The angle between the two vectors is

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}\cdot\mathbf{z}^-}{|\mathbf{n}||\mathbf{z}^-|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{149}}\right) \approx 85.3°.$$

Therefore, the angle that the tangent plane of $f(x, y) = 3x^2 - 2y$, when $x_0 = -2$ and $y_0 = 3$, makes with the *xy*-plane, is 85.3°.

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Example 27.7: A surface is defined parametrically by $\mathbf{r}(u, v) = \langle 2u + v, v - 3u, uv\rangle$. Find the equation of the tangent plane at the point $(4, -11, -6)$.

Solution: Observe that $x(u, v) = 2u + v$, $y(u, v) = v - 3u$ and $z(u, v) = uv$. From the point $(4, -11, -6)$, we can infer that $x = 2u + v = 4$ and $y = v - 3u = -11$. This is a system:

$$\begin{aligned}2u + v &= 4 \\ -3u + v &= -11.\end{aligned}$$

Solving the system, we find that $u = 3$ and $v = -2$. Note that this also checks for $z = uv = (2)(-3) = -6$.

We now need to find a vector **n** normal to the surface. Taking partial derivatives of **r**, we have

$$\mathbf{r}_u(u, v) = \langle 2, -3, v \rangle \quad \text{and} \quad \mathbf{r}_v(u, v) = \langle 1,1, u \rangle.$$

Evaluating at $u = 3$ and $v = -2$, we have

$$\mathbf{r}_u(3, -2) = \langle 2, -3, -2 \rangle \quad \text{and} \quad \mathbf{r}_v(3, -2) = \langle 1,1,3 \rangle.$$

Thus, the normal vector **n** is

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v = \langle -7, -8,5 \rangle.$$

The plane tangent to the surface $\mathbf{r}(u, v) = \langle 2u + v, v - 3u, uv \rangle$ at the point $(4, -11, -6)$ is

$$-7(x - 4) - 8\big(y - (-11)\big) + 5\big(z - (-6)\big) = 0.$$

Simplifying, we have

$$\begin{aligned} -7(x - 4) - 8(y + 11) + 5(z + 6) &= 0 \\ -7x + 28 - 8y - 88 + 5z + 30 &= 0 \\ -7x - 8y + 5z &= 30. \end{aligned}$$

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28. Differentials

The equation of the tangent plane to a point (x_0, y_0, z_0) on the surface of $z = f(x, y)$ is given by

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Add $(z - z_0)$ to both sides:

$$(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Now, view the expression $z - z_0$ as a change in z, written Δz. Do the same for $(x - x_0)$ and $(y - y_0)$. We have

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

For sufficiently small changes in the variables, we can assume that $dx \approx \Delta x$, and so on. Thus, the above equation can be written using differentials:

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy.$$

We can use this formula to study the effect that small changes in *x* and *y* have on *z*.

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Example 28.1: The exterior of a circular cylindrical tank is measured to be 4 meters in radius and 5 meters high. Assume that the measurements have a tolerance of 0.02 meters for the radius and 0.03 meters for the height. What effect do the possible variances in radius or height have on the volume of the tank?

Solution: The volume is given by $V(r, h) = \pi r^2 h$, where *r* is the radius of the base, and *h* is the vertical height. The differentials dV, dr and dh are related by the formula

$$\begin{aligned} dV &= V_r(r_0, h_0)dr + V_h(r_0, h_0)dh. \\ &= 2\pi r_0 h_0 \, dr + \pi r_0^2 \, dh \\ &= 2\pi(4)(5)(0.02) + \pi(4)^2(0.03) \\ &= 40\pi(0.02) + 16\pi(0.03). \\ &= 4.02 \text{ cubic meters.} \end{aligned}$$

It might be surprising that being off by just 0.02 meters (2 cm) when measuring the radius and 0.03 meters (3 cm) when measuring the height would translate into a change of approximately 4 cubic meters for the volume.

However, we can calculate exact volumes for these measurements and compare. If the radius is exactly 4 meters and the height exactly 5 meters, the presumptive volume is

$$V(4{,}5) = \pi(4)^2(5) = 251.327 \text{ m}^3.$$

If both measures are "low", that is, $r = 3.98$ meters and $h = 4.97$ meters, then the volume is

$$V(3.98{,}4.97) = \pi(3.98)^2(4.97) = 247.327 \text{ m}^3.$$

The difference between the two volume figures is $247.327 - 251.327 = -4 \text{ m}^3$. Thus, measuring low results in approximately 4 fewer cubic meters of volume.

If both measures are "high", $r = 4.02$ and $h = 5.03$, then the volume is

$$V(4.02{,}5.03) = \pi(4.02)^2(5.03) = 255.37 \text{ m}^3.$$

The difference between this higher figure and the presumed volume figure is $255.37 - 251.327 = 4.043 \text{ m}^3$. Again, the change in volume is roughly 4 cubic meters.

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Example 28.2: The surface area of a rectangular box of length l, width w and height h is given by $A(l, w, h) = 2(wl + wh + lh)$. Suppose workers measure the length to be 20 feet, the width 8 feet and the height 5 feet. If the tolerance of the surface area is to be no more than 6 square feet (low or high), what should the tolerances on the length, width and height be, assuming all to be the same?

Solution: Written in differential form, dA is related to dl, dw and dh by

$$\begin{aligned} dA &= A_l dl + A_w dw + A_h dh \\ &= 2(w + h)dl + 2(l + h)dw + 2(l + w)dh. \end{aligned}$$

We assume that $dl = dw = dh$. Substituting, we have

$$\begin{aligned} 6 &= 2(20 + 8)dl + 2(20 + 5)dw + 2(8 + 5)dh \\ 6 &= 56dl + 50dw + 26dh \\ 6 &= 132dl \quad (\text{since } dl = dw = dh) \end{aligned}$$

Thus, $dl = \frac{6}{132} \approx 0.045$ feet, or slightly over half an inch, in allowable tolerance. If the workers can keep their measurements for the length, width and height within this small tolerance, the actual surface area should not vary by more than 6 square feet from the presumptive surface area.

Example 28.3: A conical pyramid of sand has a circular base with radius $r = 6$ meters and a height $h = 4$ meters. If sand is added to the pile in such a way that the change in radius and the change in height are the same, what will have more of an effect on the volume, a change in the radius or a change in the height?

Solution: The volume of a conical pyramid is given by $V(r,h) = \frac{1}{3}\pi r^2 h$, where r is the radius of the base, and h is the vertical height. In differential form, we have

$$dV = \left(\frac{2}{3}\pi rh\right)\,dr + \left(\frac{1}{3}\pi r^2\right)\,dh.$$

Evaluated at $r = 6$ meters and a height $h = 4$ meters, we have

$$dV = \left(\frac{2}{3}\pi(6)(4)\right)\,dr + \left(\frac{1}{3}\pi(6)^2\right)\,dh = 16\pi\,dr + 12\pi\,dh.$$

Assuming that $dr = dh$, then since $16\pi > 12\pi$, a change in the radius will have a greater effect on the volume than an equal change in height would.

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Taking the generic differential form for $z = f(x,y)$, which is $dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$, we can divide both sides by dt, in effect forming a **related rate** in which x, y and z are functions of a parameter variable t. We use the Chain Rule and obtain:

$$\frac{dz}{dt} = f_x(x_0, y_0)\frac{dx}{dt} + f_y(x_0, y_0)\frac{dy}{dt}.$$

Example 28.4: A circular cylinder is being heated in such a way that its radius is increasing at the rate of 0.05 feet/minute and the height is shrinking at the rate of 0.02 feet/minute. Find the rate at which the surface area is changing when its base radius is 3 feet and the height is 7 feet.

Solution: Using the formula for surface area of a circular cylinder, $A(r,h) = 2\pi rh + 2\pi r^2$, we differentiate each term with respect to t:

$$\frac{dA}{dt} = A_r\frac{dr}{dt} + A_h\frac{dh}{dt} = (2\pi h + 4\pi r)\frac{dr}{dt} + (2\pi r)\frac{dh}{dt}.$$

Substituting, we have

$$\frac{dA}{dt} = \big(2\pi(7) + 4\pi(3)\big)(0.05) + \big(2\pi(3)\big)(-0.02) \approx 3.71\ \frac{\text{feet}^2}{\text{minute}}.$$