51. General Surface Integrals

The area of a surface S in R^3 defined parametrically by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ over a region of integration R in the input-variable plane is given by

$$\iint_S dS = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

Now, let $w = f(x,y,z)$ be a function defined over this surface. We then wish to calculate the **surface integral**, where $f\big(\mathbf{r}(u,v)\big) = f\big(x(u,v), y(u,v), z(u,v)\big)$:

$$\iint_S f\big(\mathbf{r}(u,v)\big)\, dS = \iint_R f\big(\mathbf{r}(u,v)\big)|\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

When the surface S is defined explicitly by a function $z = g(x,y)$, then $\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle$, and the surface integral can be rewritten

$$\iint_S f(x,y,z)\, dS = \iint_R f\big(x,y,g(x,y)\big)\sqrt{\big(g_x(x,y)\big)^2 + \big(g_y(x,y)\big)^2 + 1}\, dA,$$

Where $dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \sqrt{\big(g_x(x,y)\big)^2 + \big(g_y(x,y)\big)^2 + 1}\, dA$.

Surface area integrals are a special case of surface integrals, where $f(x,y,z) = 1$. Surface integrals can be interpreted in many ways. Some examples are discussed at the end of this section.

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Example 51.1: Find $\iint_S (x + yz)\, dS$, where S is the surface $z = 12 - 4x - 3y$ contained in the first quadrant.

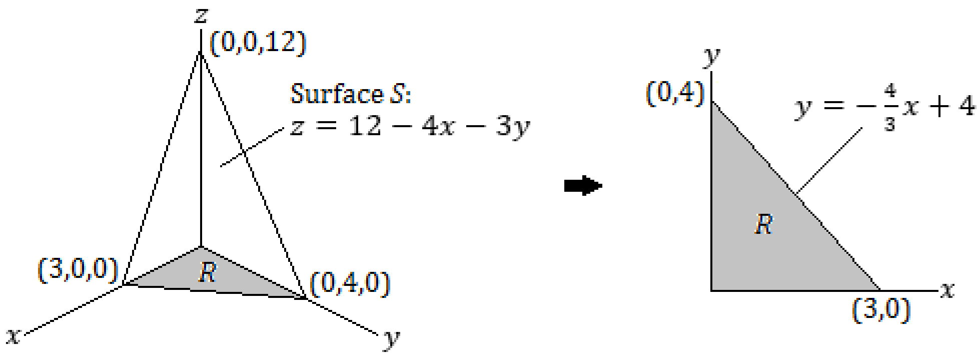
Solution: Here, $z = g(x,y) = 12 - 4x - 3y$, so that $g_x = -4$ and $g_y = -3$. Thus, dS is

$$dS = \sqrt{(-4)^2 + (-3)^2 + 1} = \sqrt{26}\, dA.$$

The integrand is written in terms of x and y, with the substitution $z = 12 - 4x - 3y$:

$$x + yz = x + y(12 - 4x - 3y) = x + 12y - 4xy - 3y^2$$

The region of integration R is the footprint of the surface S projected onto the xy-plane. Below is a sketch of S and its region of integration R. Letting $dA = dy\,dx$, we have $0 \le y \le -\frac{4}{3}x + 4$ and $0 \le x \le 3$ as the bounds of R:



The surface integral is now

$$\iint_S (x + yz)\,dS = \iint_R (x + 12y - 4xy - 3y^2)\sqrt{26}\,dA$$
$$= \sqrt{26}\int_0^3 \int_0^{-(4/3)x+4} (x + 12y - 4xy - 3y^2)\,dy\,dx.$$

The inside integral is

$$\int_0^{-(4/3)x+4} (x + 12y - 4xy - 3y^2)\,dy = \left[xy + (6 - 2x)y^2 - y^3\right]_0^{-(4/3)x+4}$$

Note that the two middle terms, $12y - 4xy$, can be written $(12 - 4x)y$, which gives $(6 - 2x)y^2$ after integration with respect to y. Substituting and simplifying, we obtain

$$x\left(-\frac{4}{3}x + 4\right) + (6 - 2x)\left(-\frac{4}{3}x + 4\right)^2 - \left(-\frac{4}{3}x + 4\right)^3 = -\frac{32}{27}x^3 + \frac{28}{3}x^2 - 28x + 32.$$

This is now integrated with respect to x:

$$\sqrt{26}\int_0^3 \left(-\frac{32}{27}x^3 + \frac{28}{3}x^2 - 28x + 32\right)dx = \sqrt{26}\left[-\frac{8}{27}x^4 + \frac{28}{9}x^3 - 14x^2 + 32x\right]_0^3$$
$$= 30\sqrt{26}.$$

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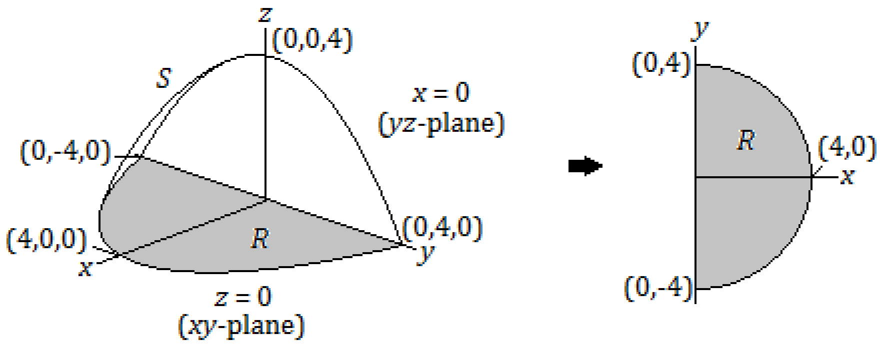
Example 51.2: Find $\iint_S x^2\, dS$, where S is the portion of sphere of radius 4, centered at the origin, such that $x \geq 0$ and $z \geq 0$.

Solution: The surface is a quarter-sphere bounded by the *xy* and *yz* planes. We sketch *S* and from it, infer the region of integration *R*:



The hemisphere can be described by rectangular coordinates $x^2 + y^2 + z^2 = 16$, in which case $z = g(x, y) = \sqrt{16 - x^2 - y^2}$. From this, we obtain partial derivatives

$$z_x = \frac{-x}{\sqrt{16 - x^2 - y^2}} \quad \text{and} \quad z_y = \frac{-y}{\sqrt{16 - x^2 - y^2}}.$$

Thus,

$$dS = \sqrt{\left(\frac{-x}{\sqrt{16 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{16 - x^2 - y^2}}\right)^2 + 1}\, dA = \frac{4}{\sqrt{16 - x^2 - y^2}}\, dA.$$

(See Example 50.3 for a similar example with all steps shown.)

Using polar coordinates, where $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$, the region R is described by $0 \leq r \leq 4$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. Due to the change of variables, the differential element is now $dA = r\, dr\, d\theta$. The surface integrals now is written

$$\iint_S x^2\, dS = \int_{-\pi/2}^{\pi/2} \int_0^4 (r\cos\theta)^2 \frac{4}{\sqrt{16 - r^2}}\, r\, dr\, d\theta.$$

This simplifies to

$$4\int_{-\pi/2}^{\pi/2}\int_{0}^{4}\left(\frac{r^3\cos^2\theta}{\sqrt{16-r^2}}\right)dr\,d\theta.$$

To integrate $\frac{r^3}{\sqrt{16-r^2}}$, we use the form taken from a table of integrals:

$$\int\frac{r^3}{\sqrt{a^2-r^2}}\,dr=-\frac{1}{3}(2a^2+r^2)\sqrt{a^2-r^2}.$$

Thus,

$$\int_0^4\frac{r^3}{\sqrt{16-r^2}}\,dr=\left[-\frac{1}{3}(32+r^2)\sqrt{16-r^2}\right]_0^4=\frac{128}{3}.$$

We now evaluate the outer integral, using the identity $\cos^2\theta=\frac{1}{2}+\frac{1}{2}\cos(2\theta)$. Note that the constant $\frac{128}{3}$ moves to the front of the integral:

$$\begin{aligned}4\left(\frac{128}{3}\right)\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta&=\frac{512}{3}\int_{-\pi/2}^{\pi/2}\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta\\&=\frac{256}{3}\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta\\&=\frac{256}{3}\left[\theta+\frac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2}\\&=\frac{256}{3}\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin(\pi)\right)-\left(-\frac{\pi}{2}+\frac{1}{2}\sin(-\pi)\right)\right]\\&=\frac{256}{3}\pi.\end{aligned}$$

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In the next example, we revisit the previous example using spherical coordinates.

Example 51.3: Find $\iint_S \; x^2\, dS$, where S is the portion of sphere of radius 4, centered at the origin, such that $x \geq 0$ and $z \geq 0$. Parameterize S using spherical coordinates.

Solution: Using spherical coordinates and the "usual" parameterization of a sphere with a fixed radius, we have

$$\mathbf{r}(\phi, \theta) = \langle 4\sin\phi\cos\theta\,, 4\sin\phi\sin\theta\,, 4\cos\phi\rangle, \text{ where } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \text{ and } 0 \leq \phi \leq \frac{\pi}{2}\,.$$

From this, we determine $\left|\mathbf{r}_\phi \times \mathbf{r}_\theta\right|$. The derivation is lengthy but not difficult, as many trigonometric identities can be used to simplify. See Example 50.4 for one such example. We have

$$\left|\mathbf{r}_\phi \times \mathbf{r}_\theta\right| = 16\sin\phi.$$

Thus, the surface integral is

$$\iint_S \; x^2\, dS = \int_{-\pi/2}^{\pi/2}\int_0^{\pi/2} (4\sin\phi\cos\theta)^2\; 16\sin\phi\; d\phi\; d\theta.$$

where $x^2 = (4\sin\phi\cos\theta)^2$ and $dS = \left|\mathbf{r}_\phi \times \mathbf{r}_\theta\right| dA = 16\sin\phi\; d\phi\; d\theta$. This simplifies to

$$256\int_{-\pi/2}^{\pi/2}\int_0^{\pi/2} \sin^3\phi\cos^2\theta\;\; d\phi\; d\theta.$$

Because the bounds are constant and the integrand held by multiplication, we can rewrite the integral as

$$256\int_{-\pi/2}^{\pi/2}\int_0^{\pi/2} \sin^3\phi\cos^2\theta\;\; d\phi\; d\theta = 256\left(\int_0^{\pi/2} \sin^3\phi\; d\phi\right)\left(\int_{-\pi/2}^{\pi/2} \cos^2\theta\; d\theta\right).$$

Both require some techniques of trigonometric integration. For the integrand $\cos^2\theta$, we use the identity $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$:

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} \cos^2\theta\; d\theta &= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta \\ &= \left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right]_{-\pi/2}^{\pi/2} \\ &= \left(\frac{\pi}{4} + 0\right) - \left(-\frac{\pi}{4} + 0\right) \\ &= \frac{\pi}{2}.\end{aligned}$$

For the integrand, $\sin^3\phi$ is rewritten as $\sin^2\phi\sin\phi = (1-\cos^2\phi)\sin\phi$:

$$\begin{aligned}\int_0^{\pi/2}\sin^3\phi\,d\phi &= \int_0^{\pi/2}\sin^2\phi\sin\phi\,d\phi\\ &= \int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\,d\phi\\ &= \int_0^{\pi/2}\sin\phi-\cos^2\phi\sin\phi\,d\phi\\ &= \left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi/2}\\ &= 0-\left(-1+\frac{1}{3}\right)\\ &= \frac{2}{3}.\end{aligned}$$

Assembling this information together, we have

$$\begin{aligned}\iint_S x^2\,dS &= 256\left(\int_0^{\pi/2}\sin^3\phi\,d\phi\right)\left(\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right)\\ &= 256\left(\frac{2}{3}\right)\left(\frac{\pi}{2}\right)\\ &= \frac{256}{3}\pi.\end{aligned}$$

You can decide if this method is more efficient than using rectangular coordinates. Clearly, both methods work.

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It is important to observe that we did *not* include the Jacobian $\rho^2\sin\phi$ when we developed the integral in the previous example (Example 51.3) in variables ϕ and θ. This is because we originally parameterized the surface in ϕ and θ, in which case, the area differential elements will always be $dA = du\,dv$, or $dA = d\phi\,d\theta$ in this case. Remember, the derivation of $dS = \left|\mathbf{r}_\phi\times\mathbf{r}_\theta\right|dA$ does not "know" whether the variables represent rectangular, spherical or cylindrical coordinate systems.

However, if we parameterize the surface in generic variables *u* and *v*, and then midway through the problem decide to integrate with respect to a different coordinate system, then we *must* include the Jacobian when we convert the area differential element. We saw this in the example prior to the last one (Example 51.2).

Setting up a surface integral is usually not difficult. However, in many cases, the integrands can be difficult to antidifferentiate. A computer, tables of integrals or numerical methods may need to be used. This is shown in the next example.

Example 51.4: Find $\iint_S x^2 z\, dS$, where *S* is the paraboloid $z = g(x, y) = 1 - x^2 - y^2$ over the *xy*-plane.

Solution: From the surface *S*, we have partial derivatives $g_x = -2x$ and $g_y = -2y$. Thus,

$$dS = \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA = \sqrt{4x^2 + 4y^2 + 1}\, dA.$$

The surface integral is

$$\iint_S x^2 z\, dS = \iint_R x^2(1 - x^2 - y^2)\sqrt{4x^2 + 4y^2 + 1}\, dA.$$

This is a difficult integral to evaluate if we remain in rectangular coordinates. Thus, we convert to polar coordinates, where the region of integration *R* is a circle of radius 1, centered at the origin on the *xy*-plane:

$$\begin{aligned}\iint_R x^2(1 - x^2 - y^2)\sqrt{4x^2 + 4y^2 + 1}\, dA &= \int_0^{2\pi}\int_0^1 (r\cos\theta)^2(1 - r^2)\sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 \cos^2\theta\,(r^3 - r^5)\sqrt{4r^2 + 1}\, dr\, d\theta,\end{aligned}$$

Using a computer or numerical methods,

$$\int_0^1 (r^3 - r^5)\sqrt{4r^2 + 1}\, dr \approx 0.143,$$

Meanwhile, $\int_0^{2\pi} \cos^2\theta\ d\theta = \pi$ (using a trigonometric identity such as in the previous example). Thus,

$$\iint_S x^2 z\, dS \approx 0.143\pi.$$

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Applications of Surface Integrals

There are a handful of common applications of surface integrals that may help one intuitively understand them better. For example, if $z = g(x, y)$ is a surface S with uniform thickness, and $f(x, y, z)$ represents the density at each point (x, y, z) on the surface, then $\iint_S f(x, y, z)\, dS$ can be interpreted as the *total mass* of S.

From Example 51.1, we had $\iint_S (x + yz)\, dS$, where S was the surface $z = 12 - 4x - 3y$ contained in the first quadrant. If x, y and z are measured in meters, and $f(x, y, z) = x + yz$ is the density of the object at the point (x, y, z) in kilograms per square meter, then $\iint_S (x + yz)\, dS$ is the total mass of the surface, in kilograms. We could interpret the result by claiming that this surface has a total mass of $30\sqrt{26}$ kilograms.

In Example 51.2, suppose that the integrand $f(x, y, z) = x^2$ represents the density of a population of bacteria (in thousands per square centimeter) on the surface at any given point (x, y, z), where the variables are measured in centimeters. Then, the total population would be given by $\iint_S x^2\, dS = \frac{256}{3}\pi \approx 268$, or about 268,000 bacteria.

Furthermore, the *average density* of the object represented by the surface would be its total mass divided by the surface's area:

$$\text{Average density} = \frac{\iint_S f(x, y, z)\, dS}{\iint_S dS}.$$

In Example 51.1, the surface area of S is $\iint_S dS = 6\sqrt{26}$ square meters. Thus, the average density of the object is $\frac{30\sqrt{26}}{6\sqrt{26}} = 5$ kilograms per square meter. In Example 51.2, the surface area of the quarter-sphere is $\iint_S dS = \frac{1}{4}\left(\frac{4}{3}\pi(4)^3\right) = \frac{64}{3}\pi$, so the average density of the bacteria on this surface is $\frac{(256/3)\pi}{(64/3)\pi} = 4$, or 4,000 bacteria per square centimeter.

Surface integrals are also used to find the flow of material through a surface, discussed in Section 53, Flux Integrals.

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