50. Surface Area Integrals

Let $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ parametrically describe a surface S in R^3. Then its surface area over a region of integration R is given by

$$\iint_S dS = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

If the surface is defined explicitly, in of the form $z = f(x,y)$, then the surface can be parametrized as

$$\mathbf{r}(x,y) = \langle x, y, f(x,y)\rangle.$$

Its partial derivatives are

$$\mathbf{r}_x = \langle 1,0, f_x(x,y)\rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,1, f_y(x,y)\rangle.$$

The cross product is

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x(x,y), -f_y(x,y), 1\rangle,$$

and the magnitude of this cross product is

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{\left(-f_x(x,y)\right)^2 + \left(-f_y(x,y)\right)^2 + 1^2} = \sqrt{\left(f_x(x,y)\right)^2 + \left(f_y(x,y)\right)^2 + 1}.$$

Thus, in the case of a surface being described by an explicitly-defined function, the surface area of the surface S over a region of integration R is

$$\iint_S dS = \iint_R \sqrt{\left(f_x(x,y)\right)^2 + \left(f_y(x,y)\right)^2 + 1}\, dx\, dy.$$

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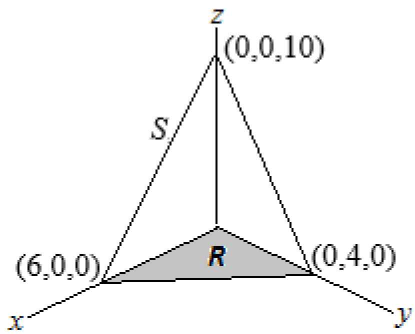
Example 50.1: Find the surface area of the plane with intercepts (6,0,0), (0,4,0) and (0,0,10) that is in the first octant.

Solution: The plane's equation is $\frac{x}{6}+\frac{y}{4}+\frac{z}{10}=1$, or $10x+15y+6z=60$. Below is a sketch of the surface *S*, the plane in the first octant, and its region of integration *R* in the *xy*-plane:



Solving for *z*, we have $z=10-\frac{5}{3}x-\frac{5}{2}y$. Therefore, the plane can be written parametrically:

$$\mathbf{r}(x,y)=\left\langle x,y,10-\frac{5}{3}x-\frac{5}{2}y\right\rangle.$$

Its partial derivatives are $\mathbf{r}_x=\left\langle 1,0,-\frac{5}{3}\right\rangle$ and $\mathbf{r}_y=\left\langle 0,1,-\frac{5}{2}\right\rangle$, and the cross product is

$$\mathbf{r}_x\times\mathbf{r}_y=\left\langle\frac{5}{3},\frac{5}{2},1\right\rangle.$$

Therefore, the magnitude is

$$\left|\mathbf{r}_x\times\mathbf{r}_y\right|=\sqrt{\left(\frac{5}{3}\right)^2+\left(\frac{5}{2}\right)^2+1^2}=\sqrt{\frac{361}{36}}=\frac{19}{6}.$$

The surface area is

$$\iint_S dS=\iint_R \left|\mathbf{r}_x\times\mathbf{r}_y\right|dA=\frac{19}{6}\iint_R dA.$$

Note that $\iint_R dA$ is the area of the region of integration *R*, which is the "shadow" cast by the plane onto the *xy*-plane, which in this case is a triangle. Using geometry, *R*'s area is $\frac{1}{2}(6)(4)=12$. Thus, the surface area of the plane $z=10-\frac{5}{3}x-\frac{5}{2}y$ in the first octant is $\frac{19}{6}(12)=38$ square units.

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Example 50.2: Find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that extends above the *xy*-plane.

Solution: The paraboloid is described parametrically by $\mathbf{r}(x, y) = \langle x, y, 9 - x^2 - y^2 \rangle$, and its partial derivatives are $\mathbf{r}_x = \langle 1,0,-2x \rangle$ and $\mathbf{r}_y = \langle 0,1,-2y \rangle$. Therefore, their cross product is

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1 \rangle,$$

and the magnitude of the cross product is

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{(2x)^2 + (2y)^2 + 1^2} = \sqrt{4x^2 + 4y^2 + 1}.$$

The paraboloid intersects the *xy*-plane ($z = 0$) at a circle of radius 3, centered at the origin, so that the region of integration R is given by $x^2 + y^2 \leq 9$. Therefore, the surface area of the paraboloid $z = 9 - x^2 - y^2$ that extends above the *xy*-plane is given by

$$\iint_S \, dS = \iint_R \, \sqrt{4x^2 + 4y^2 + 1}\, dA.$$

In rectangular coordinates, this is a difficult integrand to antidifferentiate. Instead, we use polar coordinates to rewrite this surface-area integral in terms of r and θ:

$$\iint_R \, \sqrt{4x^2 + 4y^2 + 1}\, dA = \int_0^{2\pi} \int_0^3 \sqrt{4r^2 + 1}\, r\, dr\, d\theta.$$

The inside integral is evaluated first:

$$\int_0^3 \sqrt{4r^2 + 1}\, r\, dr = \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_0^3$$
$$= \frac{1}{12}\left(37^{3/2} - 1\right).$$

Then, the outside integral is evaluated to find the surface area:

$$\frac{1}{12}\left(37^{3/2} - 1\right) \int_0^{2\pi} d\theta = \frac{\pi}{6}\left(37^{3/2} - 1\right), \text{or about } 117.32 \text{ units}^2.$$

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Example 50.3: Find the surface area of the hemisphere $x^2+y^2+z^2=25$ such that $x \geq 0$.

Solution: We can write this explicitly by solving for *x*:

$$x=f(y,z)=\sqrt{25-y^2-z^2}.$$

Thus, the hemisphere is parameterized as

$$\mathbf{r}(y,z)=\langle\sqrt{25-y^2-z^2},y,z\rangle.$$

The partial derivatives are found first:

$$\mathbf{r}_y=\left\langle-\frac{y}{\sqrt{25-y^2-z^2}},1,0\right\rangle \text{ and } \mathbf{r}_z=\left\langle-\frac{z}{\sqrt{25-y^2-z^2}},0,1\right\rangle.$$

The cross product is then determined:

$$\mathbf{r}_y\times\mathbf{r}_z=\left\langle 1,\frac{y}{\sqrt{25-y^2-z^2}},\frac{z}{\sqrt{25-y^2-z^2}}\right\rangle.$$

Then the magnitude of the cross product is determined and simplified:

$$\begin{aligned}\left|\mathbf{r}_y\times\mathbf{r}_z\right|&=\sqrt{1^2+\left(\frac{y}{\sqrt{25-y^2-z^2}}\right)^2+\left(\frac{z}{\sqrt{25-y^2-z^2}}\right)^2}\\&=\sqrt{1+\frac{y^2}{25-y^2-z^2}+\frac{z^2}{25-y^2-z^2}}\\&=\sqrt{\frac{25-y^2-z^2}{25-y^2-z^2}+\frac{y^2}{25-y^2-z^2}+\frac{z^2}{25-y^2-z^2}}\\&=\sqrt{\frac{25-y^2-z^2+y^2+z^2}{25-y^2-z^2}}\\&=\frac{5}{\sqrt{25-y^2-z^2}}.\end{aligned}$$

Thus, the surface area of the hemisphere is

$$\iint_R \frac{5}{\sqrt{25-y^2-z^2}}\,dA,$$

where *R* is the region of integration on the *yz*-plane, a circle of radius 5 centered at the origin. We rewrite this integral in terms of r and θ:

$$5\int_0^{2\pi}\int_0^5 \frac{1}{\sqrt{25-r^2}}\,r\,dr\,d\theta.$$

The inside integral is evaluated using *u-du* substitution:

$$\int_0^5 \frac{1}{\sqrt{25-r^2}}\,r\,dr = \left[-\sqrt{25-r^2}\right]_0^5 = 5.$$

Then the outer integral is evaluated:

$$5(5)\int_0^{2\pi} d\theta = 25(2\pi) = 50\pi \text{ units}^2.$$

Note that the surface area of a sphere of radius r is $A = 4\pi r^2$. Thus, the surface area of a hemisphere of radius 5 is $\frac{1}{2}(4\pi(5)^2) = 50\pi$. An alternative method of this example using spherical coordinates is presented next.

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Example 50.4: Use spherical coordinates to find the surface area of $x^2+y^2+z^2=25$ where $x \geq 0$.

Solution: Since the hemisphere lies "above" the *yz*-plane. Thus, when describing this hemisphere in spherical coordinates, the variable ϕ will be reckoned from the positive *x*-axis, such that $\phi = 0$ is the positive *x*-axis and $\phi = \frac{\pi}{2}$ is the *yz*-plane. The radius is fixed, so $\rho = 5$. The conversions are:

$$x = 5\cos\phi, \quad y = 5\sin\phi\cos\theta, \quad z = 5\sin\phi\sin\theta.$$

Thus, we can describe the parameterize the hemisphere using variables ϕ and θ:

$$\mathbf{r}(\phi,\theta) = \langle 5\cos\phi, 5\sin\phi\cos\theta, 5\sin\phi\sin\theta \rangle, \quad \text{where} \quad 0 \leq \phi \leq \frac{\pi}{2} \ \text{ and } \ 0 \leq \theta \leq 2\pi.$$

The partial derivatives are

$$\mathbf{r}_\phi = \langle -5\sin\phi, 5\cos\phi\cos\theta, 5\cos\phi\sin\theta\rangle \quad \text{and} \quad \mathbf{r}_\theta = \langle 0, -5\sin\phi\sin\theta, 5\sin\phi\cos\theta\rangle.$$

The cross product looks intimidating, but trigonometric identities will help simplify it:

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5\sin\phi & 5\cos\phi\cos\theta & 5\cos\phi\sin\theta \\ 0 & -5\sin\phi\sin\theta & 5\sin\phi\cos\theta \end{vmatrix}$$

$$= \begin{vmatrix} 5\cos\phi\cos\theta & 5\cos\phi\sin\theta \\ -5\sin\phi\sin\theta & 5\sin\phi\cos\theta \end{vmatrix}\mathbf{i} - \begin{vmatrix} -5\sin\phi & 5\cos\phi\sin\theta \\ 0 & 5\sin\phi\cos\theta \end{vmatrix}\mathbf{j} + \begin{vmatrix} -5\sin\phi & 5\cos\phi\cos\theta \\ 0 & -5\sin\phi\sin\theta \end{vmatrix}\mathbf{k}$$

$$= (25\cos\phi\sin\phi\cos^2\theta + 25\cos\phi\sin\phi\sin^2\theta)\mathbf{i} - (-25\sin^2\phi\cos\theta)\mathbf{j} + (25\sin^2\phi\sin\theta)\mathbf{k}$$

$$= (25\cos\phi\sin\phi)\mathbf{i} + (25\sin^2\phi\cos\theta)\mathbf{j} + (25\sin^2\phi\sin\theta)\mathbf{k}$$

The magnitude is found next:

$$\begin{aligned} |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{(25\cos\phi\sin\phi)^2 + (25\sin^2\phi\cos\theta)^2 + (25\sin^2\phi\sin\theta)^2} \\ &= \sqrt{625(\cos^2\phi\sin^2\phi + \sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta)} \\ &= 25\sqrt{\cos^2\phi\sin^2\phi + \sin^4\phi\,(\cos^2\theta + \sin^2\theta)} \\ &= 25\sqrt{\cos^2\phi\sin^2\phi + \sin^4\phi} \\ &= 25\sqrt{\sin^2\phi\,(\cos^2\phi + \sin^2\phi)} \\ &= 25\sqrt{\sin^2\phi} \\ &= 25\sin\phi. \end{aligned}$$

Therefore, the surface area of the hemisphere is

$$\int_0^{2\pi}\int_0^{\pi/2} 25\sin\phi\ d\phi\ d\theta.$$

The inside integral is evaluated:

$$\begin{aligned} \int_0^{\pi/2} 25\sin\phi\ d\phi &= [-25\cos\phi]_0^{\pi/2} \\ &= -25\left(\cos\frac{\pi}{2} - \cos 0\right) \\ &= -25(0-1) \\ &= 25. \end{aligned}$$

Then, the integral with respect to θ is evaluated:

$$\int_0^{2\pi} 25\ d\theta = 25(2\pi) = 50\pi.$$

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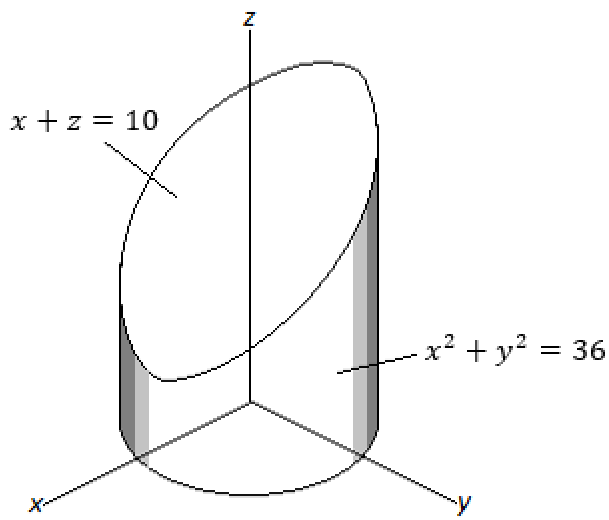
Example 50.5: A circular cylinder $x^2 + y^2 = 36$ intersects the plane $x + z = 10$. Find the surface area of this plane that is cut off by the cylinder, and then find the surface area of the cylinder that is bounded below by the *xy*-plane and above by the plane $x + z = 10$.



Solution: For the plane $x + z = 10$, we solve for z, getting $z = 10 - x$. Thus, the plane is parametrized by

$$\mathbf{r}(x, y) = \langle x, y, 10 - x \rangle.$$

Note that we use *y* as a parameter since the plane does extend into the *y* direction, even though values of *y* do not govern the values of *z*. (If it helps, think of the plane as $x + 0y + z = 10$).

The partial derivatives are $\mathbf{r}_x = \langle 1,0,-1 \rangle$ and $\mathbf{r}_y = \langle 0,1,0 \rangle$, and their cross product is

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 1,0,1 \rangle,$$

with magnitude $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$. Thus, the surface area of the plane is

$$\iint_R |\mathbf{r}_x \times \mathbf{r}_y| \, dA = \sqrt{2} \iint_R \, dA,$$

where $\iint_R \, dA$ is the area of the region of integration *R*. Since *R* is a circle of radius 6, we have $\iint_R \, dA = 36\pi$, so that the surface area of the plane is $36\pi\sqrt{2}$ units^2.

The surface area of the cylinder bounded by the xy-plane ($z = 0$) and the plane $x + z = 10$ (written $z = 10 - x = 10 - 6\cos\theta$) is found in a similar manner. First, we parametrize the cylinder:

$$\mathbf{r}(\theta, z) = \langle 6\cos\theta, 6\sin\theta, z\rangle, \quad \text{where} \quad 0 \le \theta \le 2\pi \quad \text{and} \quad 0 \le z \le 10 - 6\cos\theta.$$

The partial derivatives are $\mathbf{r}_\theta = \langle -6\sin\theta, 6\cos\theta, 0\rangle$ and $\mathbf{r}_z = \langle 0,0,1\rangle$, and their cross product is

$$\mathbf{r}_\theta \times \mathbf{r}_z = \langle 6\cos\theta, 6\sin\theta, 0\rangle.$$

The magnitude of the cross product is

$$\begin{aligned}|\mathbf{r}_\theta \times \mathbf{r}_z| &= \sqrt{(6\cos\theta)^2 + (6\sin\theta)^2 + 0^2} \\ &= \sqrt{36(\cos^2\theta + \sin^2\theta)} \\ &= \sqrt{36} \\ &= 6.\end{aligned}$$

Thus, the surface area of the cylinder is

$$\begin{aligned}\int_0^{2\pi}\int_0^{10-6\cos\theta} 6\,dz\,d\theta &= 6\int_0^{2\pi}\int_0^{10-6\cos\theta} dz\,d\theta \\ &= 6\int_0^{2\pi}(10 - 6\cos\theta)\,d\theta \\ &= 6[10\theta - 6\sin\theta]_0^{2\pi} \\ &= 120\pi \text{ units}^2.\end{aligned}$$

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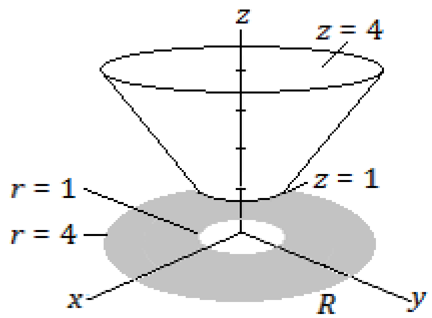
Example 50.6: Find the surface area of the cone $z = \sqrt{x^2 + y^2}$ where $1 \le z \le 4$. This is called a *band*. The solid formed by removing the apex from any conical or pyramidal object is called a *frustum*.



Solution: We have

$$\mathbf{r}(x, y) = \langle x, y, \sqrt{x^2 + y^2} \rangle.$$

We will determine the bounds of integration in a moment.

The partial derivatives are

$$\mathbf{r}_x = \langle 1,0, \frac{x}{\sqrt{x^2 + y^2}} \rangle \quad \text{and} \quad \mathbf{r}_y = \langle 0,1, \frac{y}{\sqrt{x^2 + y^2}} \rangle.$$

The cross product is

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2 + y^2}} \\ 0 & 1 & \frac{y}{\sqrt{x^2 + y^2}} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \frac{x}{\sqrt{x^2 + y^2}} \\ 1 & \frac{y}{\sqrt{x^2 + y^2}} \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & \frac{x}{\sqrt{x^2 + y^2}} \\ 0 & \frac{y}{\sqrt{x^2 + y^2}} \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k}$$

$$= -\frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j} + \mathbf{k}.$$

The magnitude is

$$
\begin{aligned}
\left|\mathbf{r}_x \times \mathbf{r}_y\right| &= \sqrt{\left(-\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(-\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1^2} \\
&= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + 1} \\
&= \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + \frac{x^2+y^2}{x^2+y^2}} \\
&= \sqrt{\frac{x^2+y^2+x^2+y^2}{x^2+y^2}} \\
&= \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}} = \sqrt{2}\,.
\end{aligned}
$$

The region of integration *R* is the area between two concentric circles, one of radius 1 and the other of radius 4. This is the "shadow" cast by the side of the conical band onto the *xy*-plane. Thus, the surface area of the band on the cone $z = \sqrt{x^2+y^2}$ where $1 \le z \le 4$ is given by

$$
\begin{aligned}
\iint_R \sqrt{2}\; dA &= \sqrt{2}\iint_R \; dA \\
&= \sqrt{2}(\pi(4)^2 - \pi(1)^2) \\
&= 15\pi\sqrt{2}\text{ units}^2.
\end{aligned}
$$

We used geometry to determine the area between the two circles, represented by $\iint_R \; dA$.

In the following example, this problem is evaluated again using cylindrical coordinates.

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Example 50.7: Use cylindrical coordinates to find the surface area of the cone $z = \sqrt{x^2 + y^2}$ where $1 \leq z \leq 4$.

Solution: In rectangular coordinates, the cone is parameterized as

$$\mathbf{r}(x, y) = \langle x, y, \sqrt{x^2 + y^2} \rangle.$$

Letting $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} \rangle = \langle r\cos\theta, r\sin\theta, r \rangle.$$

The bounds are $1 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$.

The partial derivatives are

$$\mathbf{r}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle \quad \text{and} \quad \mathbf{r}_r = \langle \cos\theta, \sin\theta, 1 \rangle.$$

The cross product is

$$\begin{aligned} \mathbf{r}_\theta \times \mathbf{r}_r &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} \\ &= \langle r\cos\theta, r\sin\theta, -r\sin^2\theta - r\cos^2\theta \rangle \\ &= \langle r\cos\theta, r\sin\theta, -r \rangle. \end{aligned}$$

The magnitude is

$$\begin{aligned} |\mathbf{r}_\theta \times \mathbf{r}_r| &= \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2 + (-r)^2} \\ &= \sqrt{r^2(\cos^2\theta + \sin^2\theta) + r^2} \\ &= \sqrt{r^2 + r^2} \\ &= r\sqrt{2}. \end{aligned}$$

The surface area is given by

$$\begin{aligned}\iint_R |\mathbf{r}_\theta \times \mathbf{r}_r|\ dA &= \int_0^{2\pi}\int_1^4 r\sqrt{2}\,dr\,d\theta\\ &= \sqrt{2}\int_0^{2\pi}\int_1^4 r\,dr\,d\theta\\ &= \sqrt{2}\int_0^{2\pi}\left[\frac{r^2}{2}\right]_1^4 d\theta\\ &= \sqrt{2}\int_0^{2\pi}\frac{15}{2}\,d\theta\\ &= \frac{15}{2}\sqrt{2}\int_0^{2\pi} d\theta\\ &= \frac{15}{2}\sqrt{2}(2\pi)\\ &= 15\pi\sqrt{2}\text{ units}^2.\end{aligned}$$

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Example 50.8: Find the surface area of $\mathbf{r}(u,v) = \langle u+v, u-v, 2uv\rangle$ over the circular region $u^2+v^2 \le 9$.

Solution: Taking partial derivatives of **r**, we have

$$\mathbf{r}_u(u,v) = \langle 1,1,2v\rangle \quad \text{and} \quad \mathbf{r}_v(u,v) = \langle 1,-1,2u\rangle.$$

Thus, $\mathbf{r}_u \times \mathbf{r}_v = \langle 2u+2v, 2v-2u, -2\rangle$, and its magnitude is

$$\begin{aligned}|\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{(2u+2v)^2+(2v-2u)^2+(-2)^2}\\ &= \sqrt{4u^2+8uv+4v^2+4v^2-8uv+4u^2+4}\\ &= \sqrt{4+8u^2+8v^2}.\end{aligned}$$

The surface area is given by

$$\iint_R |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \iint_R \sqrt{4+8u^2+8v^2}\ du\,dv.$$

Converting to polar coordinates, and noting that the region of integration is inside a circle of radius 3, we have

$$\iint_R \sqrt{4+8u^2+8v^2}\ du\,dv = \int_0^{2\pi}\int_0^3 \sqrt{4+8r^2}\,r\,dr\,d\theta.$$

The inside integral is evaluated:

$$\int_0^3 \sqrt{4+8r^2}\,r\,dr = \left[\frac{1}{24}(4+8r^2)^{3/2}\right]_0^3 = \frac{1}{24}\left(76^{3/2}-8\right).$$

The outside integral is then evaluated:

$$\begin{aligned}\int_0^{2\pi}\left(\frac{1}{24}\left(76^{3/2}-8\right)\right)d\theta &= \frac{1}{24}\left(76^{3/2}-8\right)\int_0^{2\pi} d\theta\\ &= \frac{\pi}{12}\left(76^{3/2}-8\right)\\ &\approx 171.36\ \text{units}^2.\end{aligned}$$

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The generic form of the surface-area integral (in parameters *u* and *v*), $\iint_R |\mathbf{r}_u \times \mathbf{r}_v|\,dA$, does not distinguish whether *u* and *v* are rectangular, polar, cylindrical or spherical coordinates. Thus, the area differential is always $dA = du\,dv$.

In some examples, we used a non-rectangular coordinate system to set up the integral. In such a case, we do *not* write in the usual Jacobian associated with that system. See Examples 50.4 and 50.7 for two such cases.

However, if after setting it up in a particular coordinate system, we decide to integrate it in a different coordinate system, then we must make all the necessary substitutions and then include the Jacobian. See Examples 50.2 and 50.8 for two such cases where we did include the Jacobian.