

55. Stokes Theorem

Recall that Green's Theorem allows us to find the work (as a line integral) performed on a particle around a simple closed loop path C by evaluating a double integral over the interior R that is bounded by the loop:

$$\text{Green's Theorem: } \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R (N_x - M_y) dA.$$

Green's Theorem is restricted to closed loop paths in R^2 . What about a closed loop path in R^3 ? For such paths, we use **Stokes Theorem**, which extends Green's Theorem into R^3 .

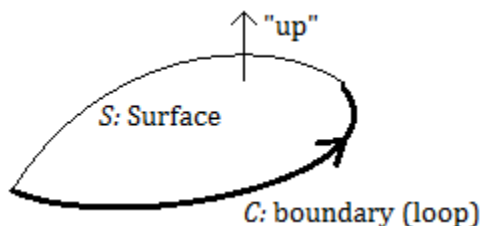
If $\mathbf{F}(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$ is a vector field and S is a simple oriented surface in R^3 with a boundary C , then Stokes Theorem is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} dS.$$

Recall that $\text{curl } \mathbf{F}$ is defined by

$$\begin{aligned} \text{curl } \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \\ &= (P_y - N_z)\mathbf{i} - (P_x - M_z)\mathbf{j} + (N_x - M_y)\mathbf{k} \\ &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle. \end{aligned}$$

A positive orientation of the surface is stated such that the path C is traversed counterclockwise. However, in R^3 , the notion of counterclockwise can be less intuitive. Thus, a positively-oriented surface is one where someone standing "up" would walk the loop with their left arm hanging over the surface.



In Green's Theorem, the surface S is the region R in the xy -plane, and "up" is in the positive z direction. Since $\mathbf{F} = \langle M, N, 0 \rangle$ in R^2 , then $\text{curl } \mathbf{F} = \langle 0, 0, N_x - M_y \rangle$ so that Green's Theorem is a special case of Stokes Theorem when limited to R^2 .

The integral $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS$ needs to be expanded so that it can be useful. Suppose for now that the surface S is defined by $z = f(x, y)$. From this, we have that \mathbf{n} is a normal vector to f by

$$\mathbf{n} = \frac{\langle f_x, f_y, -1 \rangle}{\sqrt{f_x^2 + f_y^2 + 1}} \quad \text{or} \quad \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

Also, recall that $dS = \sqrt{f_x^2 + f_y^2 + 1} \, dA$. Thus, making substitutions, we have

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_S (\text{curl } \mathbf{F}) \cdot \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{f_x^2 + f_y^2 + 1}} \sqrt{f_x^2 + f_y^2 + 1} \, dA.$$

This simplifies to

$$\iint_R (\text{curl } \mathbf{F}) \cdot \langle -f_x, -f_y, 1 \rangle \, dA.$$

The vector $\langle -f_x, -f_y, 1 \rangle$ is chosen depending on what direction "up" is stated. We may choose $\langle f_x, f_y, -1 \rangle$ in certain cases. A useful tactic is to note that if a path C is stated first, we can choose *any* surface f that is bounded by that path. Obviously, we choose "easy" surfaces in such a case.

The usual routine is as follows:

You will be asked to find the value of a line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ around a simple loop path C in R^3 . Path C may be stated explicitly, or may be implied by some surface S given by $z = f(x, y)$. You will also be given the vector field $\mathbf{F} = \langle M, N, P \rangle$.

1. Find $\text{curl } \mathbf{F}$. For now, it will be in terms of x, y and z .
2. Determine $\langle -f_x, -f_y, 1 \rangle$ or $\langle f_x, f_y, -1 \rangle$, depending on the context. Usually, the first version is used because we can always declare that positive z is "up".
3. Find $(\text{curl } \mathbf{F}) \cdot \langle -f_x, -f_y, 1 \rangle$. If variable z remains, substitute with $z = f(x, y)$. You now have an expression in terms of x and y .
4. Determine the region of integration R , which will be the footprint cast by S onto the xy -plane.
5. Integrate the result in step (3) over region R .

Normal adjustments would be made, *e.g.* if the surface was stated as $x = f(y, z)$.

Example 55.1: Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle xy, x + y + z, x^2 \rangle$ and C is a circle of radius 1, centered at the origin, in the xy -plane, traverse counterclockwise where “up” is the positive z direction.

Solution: No surface S is specified, just a boundary path C . So let’s try a couple different surfaces that have C as its boundary. First, we will let S be the interior of the circle in the xy -plane. That is, $z = f(x, y) = 0$. Thus, $\mathbf{n} = \langle 0, 0, 1 \rangle$.

Next, we find $\text{curl } \mathbf{F}$:

$$\text{curl } \mathbf{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle = \langle -1, -2x, 1 - x \rangle.$$

Thus, $(\text{curl } \mathbf{F}) \cdot \mathbf{n} = 1 - x$. This is integrated over the region inside the circle of radius 1, centered at the origin. We use polar coordinates, where $x = r \cos \theta$:

$$\begin{aligned} \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS &= \iint_S (1 - x) \, dA \\ &= \int_0^{2\pi} \int_0^1 (1 - r \cos \theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (r - r^2 \cos \theta) \, dr \, d\theta. \end{aligned}$$

We have

$$\int_0^1 (r - r^2 \cos \theta) \, dr = \left[\frac{1}{2} r^2 - \frac{1}{3} r^3 \cos \theta \right]_0^1 = \frac{1}{2} - \frac{1}{3} \cos \theta.$$

Then, we have

$$\int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{3} \cos \theta \right) d\theta = \left[\frac{1}{2} \theta - \frac{1}{3} \sin \theta \right]_0^{2\pi} = \pi.$$

Therefore, with S as the portion of the xy -plane inside the circle of radius 1 centered at the origin, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \pi.$$

Next, let's try a different surface: Let S be the paraboloid $z = f(x, y) = 1 - x^2 - y^2$ that lies above the xy -plane. Note that C is the same bounding curve. We find \mathbf{n} :

$$\mathbf{n} = \langle -f_x, -f_y, 1 \rangle = \langle -(-2x), -(-2y), 1 \rangle = \langle 2x, 2y, 1 \rangle.$$

The curl \mathbf{F} has not changed. Thus,

$$(\text{curl } \mathbf{F}) \cdot \mathbf{n} = \langle -1, -2x, 1 - x \rangle \cdot \langle 2x, 2y, 1 \rangle = -3x - 4xy + 1.$$

The region of integration is the same—the interior of the circle of radius 1, centered at the origin. Once again, we use polar coordinates:

$$\begin{aligned} \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS &= \iint_S (-3x - 4xy + 1) \, dA \\ &= \int_0^{2\pi} \int_0^1 (-3r \cos \theta - 4(r \cos \theta)(r \sin \theta) + 1) r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 (-3r^2 \cos \theta - 4r^3 \cos \theta \sin \theta + r) \, dr \, d\theta. \end{aligned}$$

The inside integral, evaluated with respect to r , is

$$\begin{aligned} \int_0^1 (-3r^2 \cos \theta - 4r^3 \cos \theta \sin \theta + r) \, dr \, d\theta &= \left[-r^3 \cos \theta - r^4 \cos \theta \sin \theta + \frac{1}{2} r^2 \right]_0^1 \\ &= -\cos \theta - \cos \theta \sin \theta + \frac{1}{2}. \end{aligned}$$

Then this is integrated with respect to θ :

$$\int_0^{2\pi} \left(-\cos \theta - \cos \theta \sin \theta + \frac{1}{2} \right) d\theta = \left[-\sin \theta - \frac{1}{2} \sin^2 \theta + \frac{1}{2} \theta \right]_0^{2\pi} = \pi.$$

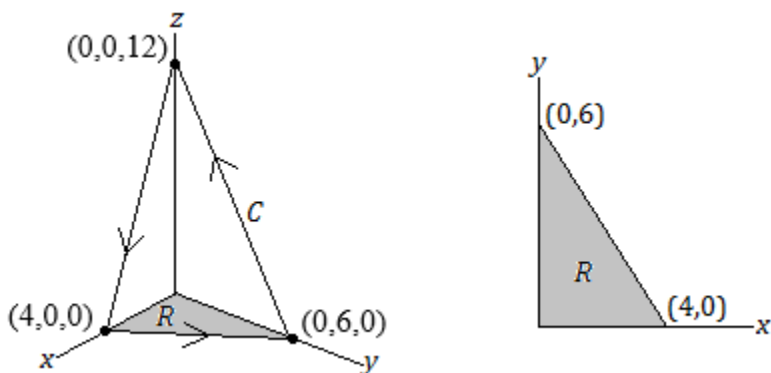
Note that at $\theta = 0$ and 2π , the sine terms vanish. Thus, we get the same result, $\int_C \mathbf{F} \cdot d\mathbf{r} = \pi$.

Try this with another surface, for example, the hemisphere of radius 1, $z = \sqrt{1 - x^2 - y^2}$.



Example 55.2: Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \langle x + y, zy, 3x \rangle$ and C is the triangle traversed from $(4,0,0)$ to $(0,6,0)$ to $(0,0,12)$, back to $(4,0,0)$. Assume “up” is in the direction of positive z .

Solution: Since no surface is specified, let’s use a plane passing through the vertices of the triangle. Below is an image of the path C and the eventual region of integration R :



The plane is $\frac{x}{4} + \frac{y}{6} + \frac{z}{12} = 1$, or $3x + 2y + z = 12$ when fractions are cleared. We can read off a normal vector from the plane’s equation: $\mathbf{n} = \langle 3, 2, 1 \rangle$. This is a useful vector since it has a 1 in the z position, agreeing with the upward direction. We find $\text{curl } \mathbf{F}$, which is $\langle -y, -3, -1 \rangle$. Thus,

$$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dS = \iint_R (-3y - 7) \, dA = \int_0^4 \int_0^{6-(3/2)x} (-3y - 7) \, dy \, dx.$$

The inside integral is

$$\begin{aligned} \int_0^{6-(3/2)x} (-3y - 7) \, dy &= \left[-\frac{3}{2}y^2 - 7y \right]_0^{6-(3/2)x} \\ &= -\frac{3}{2} \left(6 - \frac{3}{2}x \right)^2 - 7 \left(6 - \frac{3}{2}x \right) \\ &= -\frac{27}{8}x^2 + \frac{75}{2}x - 96. \end{aligned}$$

The outside integral is

$$\int_0^4 \left(-\frac{27}{8}x^2 + \frac{75}{2}x - 96 \right) \, dx = \left[-\frac{9}{8}x^3 + \frac{75}{4}x^2 - 96x \right]_0^4 = -156.$$

Therefore, $\int_C \mathbf{F} \cdot d\mathbf{r} = -156$. Let’s verify this by finding the line integral along each segment of the triangle.

From (4,0,0) to (0,6,0), we have $\mathbf{r}(t) = \langle 4 - 4t, 6t, 0 \rangle$ for $0 \leq t \leq 1$, so that $d\mathbf{r} = \langle -4, 6, 0 \rangle$. Meanwhile,

$$\begin{aligned} \mathbf{F}(t) &= \langle x + y, zy, 3x \rangle \\ &= \langle (4 - 4t) + (6t), (0)(6t), 3(4 - 4t) \rangle \quad \begin{cases} x = 4 - 4t \\ y = 6t \\ z = 0 \end{cases} \end{aligned}$$

or after simplification, $\mathbf{F}(t) = \langle 4 + 2t, 0, 12 - 12t \rangle$. Thus, $\mathbf{F} \cdot d\mathbf{r} = -4(4 + 2t) = -16 - 8t$, and the line integral is

$$\int_0^1 (-16 - 8t) dt = [-16t - 4t^2]_0^1 = -20.$$

From (0,6,0) to (0,0,12), we have $\mathbf{r}(t) = \langle 0, 6 - 6t, 12t \rangle$ for $0 \leq t \leq 1$, so that $d\mathbf{r} = \langle 0, -6, 12 \rangle$. Meanwhile, $\mathbf{F}(t) = \langle 6 - 6t, 72t - 72t^2, 0 \rangle$ after simplification.

Thus, $\mathbf{F} \cdot d\mathbf{r} = -6(72t - 72t^2) = -432(t - t^2)$, and the line integral is

$$\int_0^1 -432(t - t^2) dt = -432 \left[\frac{1}{2}t^2 - \frac{1}{3}t^3 \right]_0^1 = -72$$

From (0,0,12) to (4,0,0), we have $\mathbf{r}(t) = \langle 4t, 0, 12 - 12t \rangle$ for $0 \leq t \leq 1$. This gives $d\mathbf{r} = \langle 4, 0, -12 \rangle$. Also, $\mathbf{F}(t) = \langle 4t, 0, 12t \rangle$ after simplification. Therefore,

$$\mathbf{F} \cdot d\mathbf{r} = 4(4t) - 12(12t) = 16t - 144t = -128t.$$

Finally, the line integral is

$$\int_0^1 -128t dt = -128 \left[\frac{1}{2}t^2 \right]_0^1 = -64.$$

The sum of these three line integrals is $-20 - 72 - 64 = -156$, agreeing with the result found by Stokes Theorem.



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