40. Spherical Coordinate System

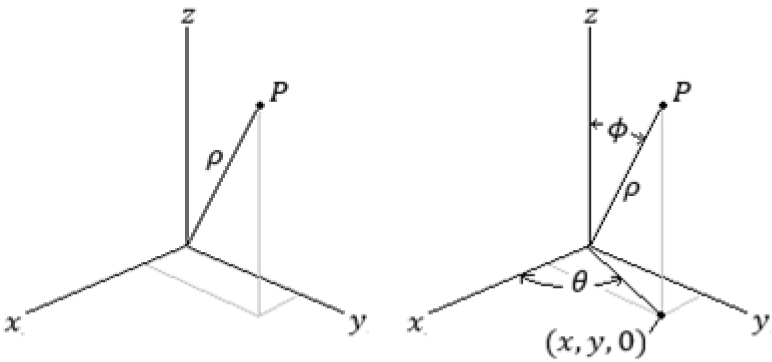
A point $P = (x, y, z)$ described by rectangular coordinates in R^3 can also be described by three independent variables, ρ (rho), θ and ϕ (phi), whose meanings are given below:

ρ: the distance from the origin to P.

θ: the angle from the positive *x*-axis to the line connecting the origin to the point $(x, y, 0)$.

ϕ: the angle from the positive *z*-axis to the line connecting the origin to P.

Descriptively, ρ (rho) is the spherical radius, θ is the "sweep" or "azimuth" angle of the point's projection onto the *xy*-plane, and ϕ (phi) is the "lean" angle of the point relative to the positive *z*-axis.



A point *P* is shown in R^3. A line connects the origin to *P* (above left). This line has length ρ. This same line also "leans" at an angle of ϕ radians, relative to the positive *z*-axis (above right). Furthermore, a line is drawn from the origin to the point's projection onto the *xy*-plane, $(x, y, 0)$. This line is at an angle of θ radians relative to the positive *x*-axis in the counterclockwise manner. The use of θ here is identical to its usage in the polar and cylindrical coordinate systems and is confined to the *xy*-plane.

These three variables comprise the **spherical coordinate system** and are best used to describe regions in R^3 that are spheres, or parts of a sphere. For such regions, the bounds of ρ, θ and ϕ will be constants. The common restrictions on ρ, θ and ϕ are:

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi.$$

Any point in R^3 can be described by spherical coordinates (ρ, θ, ϕ) that meet the restrictions stated above.

The variable ϕ can be thought of as the "lean" of the line connecting the origin to P relative to the positive z-axis. If $\phi = 0$, then P lies on the positive z-axis. If $\phi = \frac{\pi}{2}$, then P lies on the xy-plane, which is at right angles to the positive z-axis, and if $\phi = \pi$, then P lies on the negative z-axis.

The conversion formulas between rectangular coordinates (x, y, z) and spherical coordinates (ρ, θ, ϕ) are:

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \arctan\left(\frac{y}{x}\right), \quad \phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right).$$

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

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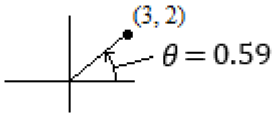
A Review of the Arctangent Operator,
or Getting your θ and ϕ angles correct!

Assume x and y are rectangular coordinates in the xy-plane, and that θ is the angle from the positive x-axis to the line connecting the origin to the point (x,y), given by $\theta = \arctan(y/x)$. When calculating the angle θ, we must be careful in handling the result as given by a calculator. Recall that the arctangent operator only returns values in the first quadrant (if y/x is positive) or fourth quadrants (if y/x is negative).

If both x and y are positive (in Quadrant 1), then y/x is positive, so that $\theta = \arctan\left(\frac{y}{x}\right)$ is in the interval $0 \le \theta \le \frac{\pi}{2}$ (the first quadrant).

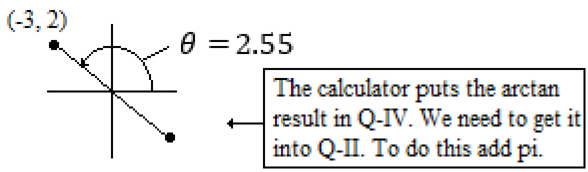
Example: $x = 3, y = 2$, so that $\theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.59$ radians.



What needs to be done: Nothing. The arctan operator returned a value in the Quadrant-I, as expected. The correct answer is $\theta = 0.59$ radians.

If x is negative and y is positive (in Quadrant II), then y/x is negative, but $\theta = \arctan(y/x)$ is in the interval $-\frac{\pi}{2} \le \theta \le 0$ (the fourth quadrant). We must add π to this result to place the angle in the interval $\frac{\pi}{2} \le \theta \le \pi$, the second quadrant.

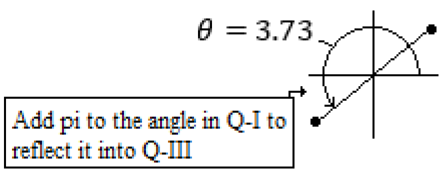
Example: $x = -3, y = 2$, so that $\theta = \tan^{-1}\left(\frac{2}{-3}\right) \approx -0.59$ radians. This result is in Quadrant-IV. We need to get it in Quadrant-II.



What needs to be done: The value –0.59 is in Quadrant -IV, so we add π to place the angle into Quadrant-II. The correct answer is $\theta = -0.59 + 3.14 = 2.55$ radians.

If both x and y are negative (in Quadrant 3), then y/x is positive, but $\theta = \arctan(y/x)$ is in the interval $0 \le \theta \le \frac{\pi}{2}$ (the first quadrant). Thus, we add π to this result to place the angle in the interval $\pi \le \theta \le \frac{3\pi}{2}$, the third quadrant.

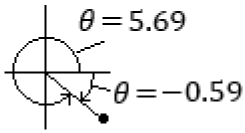
Example: $x = -3, y = -2$, so that $\theta = \tan^{-1}\left(\frac{-2}{-3}\right) = \tan^{-1}\left(\frac{2}{3}\right) \approx 0.59$. This result is in Quadrant-I. We need to get it in Quadrant-III.



What needs to be done: The value 0.59 is in Quadrant-I, so add π to place the angle into Quadrant-III. The correct answer is $\theta = 0.59 + 3.14 = 3.73$ radians.

If x is positive and y is negative (in Quadrant 4), then y/x is negative, so that $\theta = \arctan(y/x)$ is in the interval $-\frac{\pi}{2} \leq \theta \leq 0$ (the fourth quadrant). This is usually acceptable. However, if we desire that the angle be positive, then we add 2π to the result.

Example: $x = 3, y = -2$, so that $\theta = \tan^{-1}\left(\frac{-2}{3}\right) \approx -0.59$. This result is inn Quadrant 4.



What needs to be done: Nothing. The value is in quadrant IV, as expected. However, if the angle is to be stated as a positive number, add 2π. Thus, $\theta = -0.59 + 6.28 = 5.69$ radians.

For ϕ, the process is simpler. If the point lies above the *xy*-plane (that is, *z* is positive), then the result given by $\phi = \arctan\left(\sqrt{x^2 + y^2}/z\right)$ will be in the interval $0 \leq \phi \leq \frac{\pi}{2}$ and no adjustments need to be made. If *z* is negative, then ϕ must be in the interval $\frac{\pi}{2} \leq \phi \leq \pi$. However, the expression $\arctan\left(\sqrt{x^2 + y^2}/z\right)$ will give a value in $-\frac{\pi}{2} \leq \phi \leq 0$, in which case, add π to place the angle ϕ in the desired interval of $\frac{\pi}{2} \leq \phi \leq \pi$.

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Example 40.1: Convert the rectangular coordinate (2,5,3) into spherical coordinates.

Solution: Note that this point lies above the first quadrant of the *xy*-plane. Thus, we expect that both θ and ϕ will be in the intervals $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$. We have

$$\rho = \sqrt{2^2 + 5^2 + 3^2} = \sqrt{38},$$
$$\theta = \arctan\left(\frac{5}{2}\right) \approx 1.1903 \text{ radians},$$
$$\phi = \arctan\left(\frac{\sqrt{2^2 + 5^2}}{3}\right) \approx 1.0625 \text{ radians}.$$

Since $\frac{\pi}{2} \approx 1.571$, the values for θ and ϕ are plausible.

Example 40.2: Convert the rectangular coordinate $(-3, -4, -2)$ into spherical coordinates.

Solution: This point lies below the third quadrant of the *xy*-plane. We expect that θ will be in the interval $\pi < \theta < \frac{3\pi}{2}$ and that ϕ will be in the interval $\frac{\pi}{2} < \phi < \pi$. We have

$$\rho = \sqrt{(-3)^2 + (-4)^2 + (-2)^2} = \sqrt{29},$$
$$\theta = \arctan\left(\frac{-4}{-3}\right) = \arctan\left(\frac{4}{3}\right) \approx 0.9273 \text{ radians},$$
$$\phi = \arctan\left(\frac{\sqrt{(-3)^2 + (-4)^2}}{-2}\right) = \arctan\left(-\frac{5}{2}\right) \approx -1.1903 \text{ radians}.$$

The current value for θ is incorrect. The value of 0.9273 radians places θ in the first quadrant. Thus, add π, so that the correct value for θ is 0.9273 + 3.1416 ≈ 4.0689 radians, which is in the in the interval $\pi < \theta < \frac{3\pi}{2}$, as desired.

Furthermore, we can rewrite ϕ so that it is in the interval $\frac{\pi}{2} < \phi < \pi$. We add π to $\phi \approx -1.1903$, getting –1.1903 + 3.1416 ≈ 1.9513 radians, which is an angle in the desired interval.

To summarize, the point $(-3, -4, -2)$ in rectangular coordinates is equivalent to the point $(\rho, \theta, \phi) = (\sqrt{29}, 4.0689, 1.9513)$ in spherical coordinates.

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Example 40.3: Describe the solid sphere of radius 2 centered at the origin using spherical coordinates.

Solution: The solid sphere of radius 2 is described by

$$0 \le \rho \le 2, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

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Example 40.4: Describe the solid hemisphere of radius 4, bounded by the *xy*-plane, extending into the negative *z* direction.

Solution: We have

$$0 \le \rho \le 4, \quad 0 \le \theta \le 2\pi, \quad \frac{\pi}{2} \le \phi \le \pi.$$

Note that the bounds $\frac{\pi}{2} \leq \phi \leq \pi$ indicate that points in this region lie below the *xy*-plane.

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Example 40.5: Describe $\rho = 3$, with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

Solution: This is a sphere of radius 3, centered at the origin. Had we set $0 \leq \rho \leq 3$, this would describe the solid sphere of radius 3.

Converting back to rectangular coordinates, this same spherical surface is given by

$$x = 3\sin\phi\cos\theta$$
$$y = 3\sin\phi\sin\theta$$
$$z = 3\cos\phi,$$

with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

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Example 40.6: Describe the solid given by $9 \leq x^2 + y^2 + z^2 \leq 25$, where $x \geq 0$ and $y \geq 0$, using spherical coordinates.

Solution: Note that *x* and *y* are restricted to the first quadrant in the *xy*-plane, so that θ cannot be greater than $\frac{\pi}{2}$. The object is two nested spheres, one of radius 3 and the other of radius 5, lying above and below the first quadrant of the *xy*-plane. Note that *z* still may be positive or negative. We have $3 \leq \rho \leq 5, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq \pi$.

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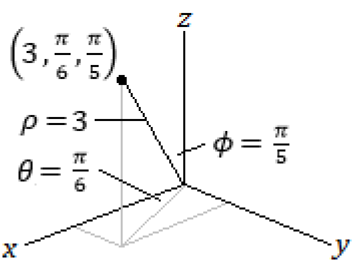
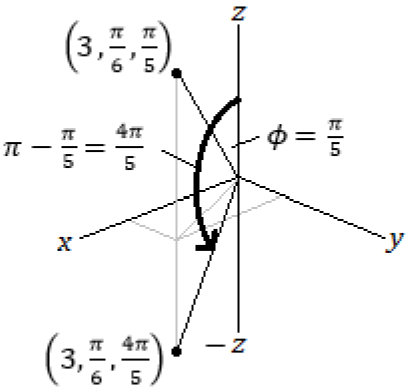
Example 40.7: Given the point *P* defined by spherical coordinates $(\rho, \theta, \phi) = \left(3, \frac{\pi}{6}, \frac{\pi}{5}\right)$, find the reflection of *P* (a) cross the *xy*-plane, (b) across the *yz*-plane, and (c) across the *xz*-plane.

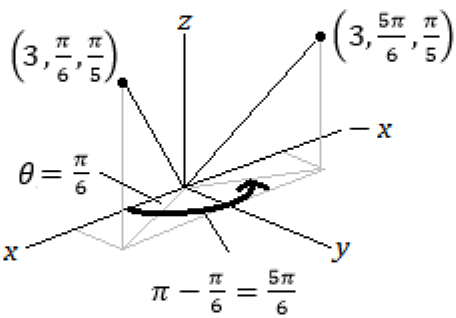


Solution: Unlike the rectangular coordinate axis system, where we can negate certain values within the ordered triple to achieve a reflection, we must remember that the point in spherical coordinates is partially described by angle measures.

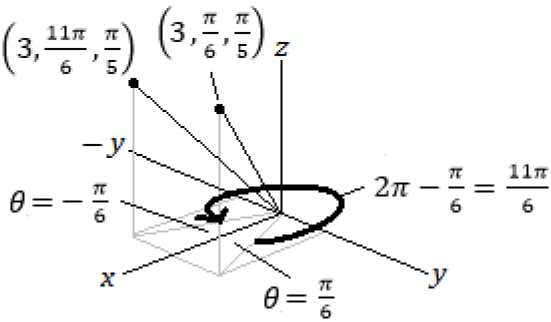
a) When P is reflected across the *xy*-plane, the ρ and θ values do not change. However, the new ϕ value is now the supplement of the original value. Thus, the reflection of P across the *xy*-plane is given by $\left(3, \frac{\pi}{6}, \pi - \frac{\pi}{5}\right) = \left(3, \frac{\pi}{6}, \frac{4\pi}{5}\right)$.



b) When P is reflected across the *yz*-plane, the ρ and ϕ values do not change. However, the new θ value is now the supplement of the original value. Thus, the reflection of P across the *yz*-plane is given by $\left(3, \pi - \frac{\pi}{6}, \frac{\pi}{5}\right) = \left(3, \frac{5\pi}{6}, \frac{\pi}{5}\right)$.



c) When P is reflected across the *xz*-plane, the ρ and ϕ values do not change. However, the new θ value is now the negation of the original value. Thus, the reflection of P across the *xz*-plane is given by $\left(3, -\frac{\pi}{6}, \frac{\pi}{5}\right)$. If we require that θ be positive, then this point is also described by $\left(3, 2\pi - \frac{\pi}{6}, \frac{\pi}{5}\right) = \left(3, \frac{11\pi}{6}, \frac{\pi}{5}\right)$.



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Example 40.8: Rewrite the point P given by spherical coordinates $(\rho, \theta, \phi) = \left(-3, \frac{\pi}{6}, \frac{\pi}{5}\right)$ so that all values are positive.

Solution: Note that the angles $\theta = \frac{\pi}{6}$ and $\phi = \frac{\pi}{5}$ describe a ray extending from the origin into the positive octant, where x, y and z are all positive. Any point on this ray would have a positive value for ρ, being the point's distance from the origin.

If the ray is extended in the opposite direction, it extends into the octant where x, y and z are all negative. Thus, a "distance" of $\rho = -3$ is interpreted as a point that lies on this ray such that its x, y and z coordinates would all be negative

We adjust the angle values so that $\pi \le \theta \le \frac{3\pi}{2}$, making both the x and y coordinates negative, and $\frac{\pi}{2} \le \phi \le \pi$, making the z coordinate negative. For θ, we add π to the original angle measure, and for ϕ, we use the supplement of the original angle measure. Thus, $\left(-3, \frac{\pi}{6}, \frac{\pi}{5}\right)$ is equivalent to $\left(-3, \pi + \frac{\pi}{6}, \pi - \frac{\pi}{5}\right) = \left(3, \frac{7\pi}{6}, \frac{4\pi}{5}\right)$.

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41. Integration with Spherical Coordinates

A function $f(x,y,z)$ integrated over a region *R* can be integrated in spherical coordinates, where $\rho^2 \sin\phi$ is the Jacobian, and present in all integrals defined in spherical coordinates.

$$\iiint_R f(x,y,z)\,dV$$
$$= \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \int_{\rho_1}^{\rho_2} f\big(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi)\big)\ \rho^2 \sin\phi\ d\rho\, d\phi\, d\theta.$$

Here, $f\big(x(\rho,\theta,\phi), y(\rho,\theta,\phi), z(\rho,\theta,\phi)\big)$ represents the integrand after the variables *x*, *y* and *z* have been converted into spherical coordinates.

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Example 41.1: Use spherical coordinates to find the volume of a sphere of radius 1.

Solution. The sphere is described in spherical coordinates by

$$0 \le \rho \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

The integrand is $f(x,y,z) = 1$, since this is a volume integral in R^3. The volume element is

$$dV = \rho^2 \sin\phi\ d\rho\, d\phi\, d\theta.$$

Thus, we have

$$\iiint_R 1\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} 1\, \rho^2 \sin\phi\ d\rho\ d\phi\ d\theta.$$

The inner-most integral is evaluated first with respect to ρ:

$$\int_0^1 \rho^2 \sin\phi\ d\rho = \sin\phi \int_0^1 \rho^2\, d\rho$$
$$= \sin\phi \left[\frac{\rho^3}{3}\right]_0^1$$
$$= \frac{1}{3}\sin\phi.$$

This is then integrated with respect to ϕ:

$$\int_0^{\pi}\left(\frac{1}{3}\sin\phi\right)\,d\phi = \frac{1}{3}\int_0^{\pi}\sin\phi\,\,d\phi$$
$$= \frac{1}{3}[-\cos\phi]_0^{\pi}$$
$$= \frac{1}{3}[(-\cos\pi)-(-\cos 0)] \quad \begin{cases} -\cos\pi = -(-1) = 1 \\ -\cos 0 = -1 \end{cases}$$
$$= \frac{2}{3}.$$

Finally, we integrate with respect to θ:

$$\int_0^{2\pi}\left(\frac{2}{3}\right)\,d\theta = \frac{2}{3}(2\pi) = \frac{4\pi}{3}.$$

From geometry, we know that the volume of a sphere of radius 1 is $\frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$. This is a check our work.

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Example 41.2: Evaluate $\iiint_R \sqrt{x^2+y^2+z^2}\,dV$, where R is a hemisphere of radius 5, centered at the origin and above the *xy*-plane.

Solution: In rectangular coordinates, the triple integral is

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_0^{\sqrt{25-x^2-y^2}}\sqrt{x^2+y^2+z^2}\,dz\,\,dy\,\,dz.$$

In spherical coordinates, the integrand is rewritten as $\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$, then multiplied by the Jacobian $\rho^2\sin\phi$. This same integral in spherical coordinates is

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{5}(\rho)\,\rho^2\sin\phi\,\,d\rho\,\,d\phi\,\,d\theta = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{5}\rho^3\sin\phi\,\,d\rho\,\,d\phi\,\,d\theta.$$

This integral has constant bounds and is more easily solved using spherical coordinates than in rectangular coordinates.

The inner-most integral is evaluated first with respect to ρ:

$$\int_0^5 \rho^3 \sin\phi \; d\rho = \sin\phi \int_0^5 \rho^3 \; d\rho$$
$$= \sin\phi \left[\frac{\rho^4}{4}\right]_0^5$$
$$= \frac{625}{4}\sin\phi.$$

Then, this is then integrated with respect to ϕ:

$$\int_0^{\pi/2} \left(\frac{625}{4}\sin\phi\right) \; d\phi = \frac{625}{4}[-\cos\phi]_0^{\pi/2}$$
$$= \frac{625}{4}\big(0-(-1)\big)$$
$$= \frac{625}{4}.$$

Lastly, we integrate with respect to θ:

$$\int_0^{2\pi} \left(\frac{625}{4}\right) \; d\theta = \frac{625}{4}(2\pi) = \frac{625\pi}{2}.$$

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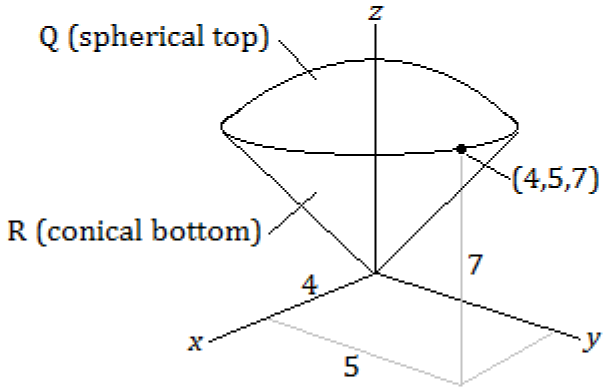
Comment: When using spherical coordinates to find the volume of a solid in R^3 (or any situation where the variables in the integrand can be isolated as separate factors) and assuming all bounds are constants, then the triple integral can be written as the product of three single integrals:

$$\text{Volume} = \int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta$$
$$= \left(\int_{\theta_1}^{\theta_2} d\theta\right)\left(\int_{\phi_1}^{\phi_2} \sin\phi \; d\phi\right)\left(\int_{\rho_1}^{\rho_2} \rho^2 \; d\rho\right).$$

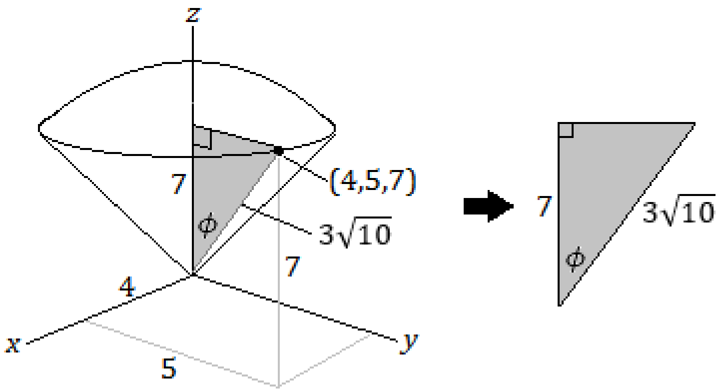
Example 41.3: Let Q be a sphere centered at the origin, and R be a cone whose vertex is at the origin and opens in the positive z direction. The solid S bounded inside the cone and the sphere is called a *spherical sector*. Suppose the point (4,5,7) in rectangular coordinates lies on the "lip", where the sphere and the cone intersect. Find the volume of S.

Solution: We can determine bounds for ρ, θ and ϕ by sketching the solid and the point on its rim:



The distance from (0,0,0) to (4,5,7) is $\sqrt{4^2+5^2+7^2} = \sqrt{90} = 3\sqrt{10}$. Since the solid includes the origin, the bounds of ρ are $0 \le \rho \le 3\sqrt{10}$.

Note that the solid includes the positive z-axis, so the lower bound for ϕ is 0. The upper bound is found by observing a right triangle with the adjacent leg on the z-axis, and the hypotenuse corresponding to a line from the origin to the point (4,5,7). From this, we see that for an upper bound, we have $\phi = \arccos\left(\frac{7}{3\sqrt{10}}\right)$. Lastly, the solid encircles the z-axis. Thus, the bounds of θ are $0 \le \theta \le 2\pi$.



The volume integral in spherical coordinates is

$$\int_0^{2\pi} \int_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)} \int_0^{3\sqrt{10}} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta.$$

The inner-most integral is evaluated with respect to ρ:

$$\begin{aligned}\int_0^{3\sqrt{10}} \rho^2 \sin\phi \; d\rho &= \sin\phi \int_0^{3\sqrt{10}} \rho^2 \; d\rho \\ &= \sin\phi \left[\frac{\rho^3}{3}\right]_0^{3\sqrt{10}} \\ &= \frac{\left(3\sqrt{10}\right)^3}{3} \sin\phi \\ &= 9(10)^{3/2} \sin\phi.\end{aligned}$$

This is then integrated with respect to ϕ:

$$\begin{aligned}\int_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)} \left(9(10)^{3/2} \sin\phi\right) d\phi &= 9(10)^{3/2} \int_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)} \sin\phi \; d\phi \\ &= 9(10)^{3/2} [-\cos\phi]_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)} \\ &= 9(10)^{3/2} \left[-\cos\left(\arccos\left(\frac{7}{3\sqrt{10}}\right)\right) - (-\cos 0)\right] \\ &= 9(10)^{3/2} \left(1 - \frac{7}{3\sqrt{10}}\right).\end{aligned}$$

Finally, we evaluate the outer-most integral with respect to θ:

$$\begin{aligned}\int_0^{2\pi} 9(10)^{3/2} \left(1 - \frac{7}{3\sqrt{10}}\right) \; d\theta &= 9(10)^{3/2} \left(1 - \frac{7}{3\sqrt{10}}\right) 2\pi \\ &= 18\pi(10)^{3/2} \left(1 - \frac{7}{3\sqrt{10}}\right) \\ &\approx 468.76 \text{ cubic units.}\end{aligned}$$

Using the short form

$$\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2}\rho^2\sin\phi\ d\rho\, d\phi\, d\theta = \left(\int_{\theta_1}^{\theta_2} d\theta\right)\left(\int_{\phi_1}^{\phi_2}\sin\phi\ d\phi\right)\left(\int_{\rho_1}^{\rho_2}\rho^2\ d\rho\right),$$

we have

$$\begin{aligned}\text{Volume} &= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)}\sin\phi\ d\phi\right)\left(\int_0^{3\sqrt{10}}\rho^2\ d\rho\right)\\ &= \left([\theta]_0^{2\pi}\right)\left([-\cos\theta]_0^{\arccos\left(\frac{7}{3\sqrt{10}}\right)}\right)\left(\left[\frac{\rho^3}{3}\right]_0^{3\sqrt{10}}\right)\\ &= (2\pi)\left(-\frac{7}{3\sqrt{10}}-(-1)\right)\left(\frac{\left(3\sqrt{10}\right)^3}{3}\right)\\ &= 9(10)^{3/2}\left(1-\frac{7}{3\sqrt{10}}\right)2\pi\\ &= 18\pi(10)^{3/2}\left(1-\frac{7}{3\sqrt{10}}\right)\approx 468.76 \text{ cubic units.}\end{aligned}$$

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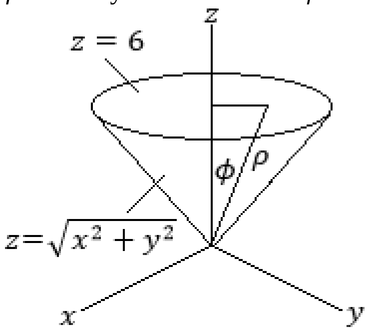
Example 41.4: Use spherical coordinates to find the volume contained within the cone $z=\sqrt{x^2+y^2}$ and below the plane $z=6$.

Solution: First, observe that the solid is not a spherical sector as in the previous example. The value of ρ will vary as a function of ϕ.



Note that ρ will vary as a function of ϕ.

The bounds for θ and ϕ are easy to determine. The "sweep" angle θ encompasses a full counter-clockwise rotation around the *xy*-plane from the positive *x*-axis back to the positive *x*-axis, so that $0 \leq \theta \leq 2\pi$. The "lean" angle ϕ varies from 0 (the positive *z*-axis) to $\frac{\pi}{4}$ (the side of the cone, which is 45 degrees from both the positive *x*-axis and the positive *y*-axis).

For the plane $z = 6$, substitute $z = \rho \cos \phi$, getting $\rho \cos \phi = 6$. Then solving for ρ, we have $\rho = 6/\cos \phi = 6 \sec \phi$. Since the object is a solid and includes the origin, the lower bound for ρ is 0, while the upper bound is the plane, so that the bounds for ρ are $0 \leq \rho \leq 6 \sec \phi$. Thus, the volume integral is

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{6 \sec \phi} \rho^2 \sin \phi \; d\rho \, d\phi \, d\theta.$$

The inner-most integral is integrated with respect to ρ:

$$\begin{aligned} \int_0^{6 \sec \phi} \rho^2 \sin \phi \; d\rho &= \sin \phi \left[\frac{\rho^3}{3} \right]_0^{6 \sec \phi} \\ &= \sin \phi \left(\frac{216 \sec^3 \phi}{3} \right) \\ &= 72 \sin \phi \cos^{-3} \phi. \end{aligned}$$

This is now integrated with respect to ϕ. Note that $72 \sin \phi \cos^{-3} \phi$ can be antidifferentiated by a *u*-*du* substitution, where $u = \cos \phi$ so that $du = -\sin \phi \; d\phi$. This results in a power-rule form, $\int (-72u^{-3}) \, du = 36u^{-2}$:

$$\begin{aligned} \int_0^{\pi/4} 72 \sin \phi \cos^{-3} \phi \; d\phi &= [36 \cos^{-2} \phi]_0^{\pi/4} \\ &= 36 \left(\frac{\sqrt{2}}{2} \right)^{-2} - (36(1)^{-2}) \\ &= 36(2) - 36 \\ &= 36\,. \end{aligned}$$

Lastly, we integrate with respect to θ:

$$\int_0^{2\pi} (36) \, d\theta = 36(2\pi) = 72\pi \text{ cubic units.}$$