33. Riemann Summation over Rectangular Regions

A rectangular region *R* in the *xy*-plane can be defined using compound inequalities, where *x* and *y* are each bound by constants such that $a_1 \le x \le a_2$ and $b_1 \le y \le b_2$. Let z = f(x, y) be a continuous function defined over a rectangular region *R* in the *xy*-plane. The notation

$$\iint_R^{\square} f(x,y) \, dA$$

represents the **double integral** of z = f(x, y) over *R*. The *dA* represents "area element", and is either *dy dx* or *dx dy*. Thus, we can write

$$\iint_{R}^{\square} f(x,y) \, dA = \int_{a_{1}}^{a_{2}} \int_{b_{1}}^{b_{2}} f(x,y) \, dy \, dx = \int_{b_{1}}^{b_{2}} \int_{a_{1}}^{a_{2}} f(x,y) \, dx \, dy.$$

Note that the bounds a_1 and a_2 correspond with the differential dx, and bounds b_1 and b_2 correspond with dy.

The value of a double integral can be approximated by **Riemann sums** adapted to the two-dimensional case. Interval $a_1 \le x \le a_2$ is subdivided into msubdivisions (not necessarily of equal size) and interval $b_1 \le y \le b_2$ is subdivided into n subdivisions (again, not necessarily of equal size). If we define indices $1 \le i \le m$ and $1 \le j \le n$, then we have a way to identify a particular subdivision within region R. For example, if $a_1 \le x \le a_2$ is subdivided into 4 subdivisions and $b_1 \le y \le b_2$ is subdivided into 5 subdivisions, then (x_2, y_3) is a representative point within the 2nd subdivision of the *x*-interval and the 3rd subdivision of the *y*-interval, and $f(x_2, y_3)$ is the function evaluated at (x_2, y_3) .

Using this scheme, a double integral can be approximated by a double sum over *i* and *j*:

$$\iint_{R}^{\Box} f(x,y) \, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{i},y_{j}) \, \Delta y \, \Delta x \text{ or } \sum_{j=1}^{n} \sum_{i=1}^{m} f(x_{i},y_{j}) \, \Delta x \, \Delta y.$$

Example 33.1: Use Riemann Sums to approximate $\iint_{R}^{\square} x^2 y \, dA$ where *R* is the rectangle $0 \le x \le 3$ and $1 \le y \le 5$ in the *xy* plane. Subdivide the region *R* into subregions each with length 1 to a side, and from each subregion, choose *x* and *y* to be the "upper right" corner.

Solution: The rectangular region *R* is shown at right, subdivided into subregions, so that $\Delta A = \Delta x \Delta y = (1)(1) = 1$. There are 12 such subregions.

Then choose a representative point (x_i, y_j) within each subregion. In this example, we choose (x_i, y_j) to be the "upper right" point within each subregion (this is an arbitrary choice. We could choose the "lower left" or the "middle point", and so on). Here, $1 \le i \le 3$ and $2 \le j \le 5$, the bounds chosen for convenience.

Next, evaluate the integrand $z = f(x, y) = x^2 y$ at the representative points (x_i, y_j) :

$$f(1,5) = 5 \quad f(2,5) = 20 \quad f(3,5) = 45$$

$$f(1,4) = 4 \quad f(2,4) = 16 \quad f(3,4) = 36$$

$$f(1,3) = 3 \quad f(2,3) = 12 \quad f(3,3) = 27$$

$$f(1,2) = 2 \quad f(2,2) = 8 \quad f(3,2) = 18$$

Visually, we have a surface $z = f(x, y) = x^2 y$ "above" the *xy*-plane. Each subregion in *R* is the base of a rectangular box whose height is the function value shown in the table above. Each box has a volume of $f(x_i, y_j) dA$. Since dA = dx dy = (1)(1) = 1 in each case, each box has volume $f(x_i, y_j) \times 1$, or simply $f(x_i, y_j)$. The value of $\iint_R^{\square} x^2 y dA$ is approximated by the sum of the volumes of the rectangular boxes contained within it. Thus,

$$\iint_{R}^{\square} x^{2} y \, dA \approx \sum_{i=1}^{3} \sum_{j=2}^{5} f(x_{i}, y_{j}) \, \Delta y \, \Delta x$$

= 2 + 8 + 18 + 3 + 12 + 27 + 4 + 16 + 36 + 5 + 20 + 45
= 196.





Note that if we chose the representative point to be the lower-left corner of each subregion, we would find that the Riemann Sum is 50. The mean, $\frac{196+50}{2} = 123$, is a reasonable approximation of $\iint_{R}^{\square} x^2 y \, dA$.

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Example 33.2: Use Riemann Sums to approximate $\iint_R^{\square} g(x, y) dA$, where *g* is shown by the contour map below. Let the region of integration *R* be given by $-4 \le x \le 4, -6 \le y \le 6$, and let $\Delta x = 2$ and $\Delta y = 2$. Use the middle point within each subregion.



Solution: The region *R* is identified and then subdivided into 2×2 subregions (lower left, boldfaced). Then the middle point (x_i, y_j) from within each subregion is identified (lower right):



The values of z = g(x, y) are estimated from the contour map. For example, in the top tier of subregions, reading left to right and using the middle points, the values of *g* are approximately g(-3,5) = 37, g(-1,5) = 46, g(1,5) = 55 and g(3,5) = 60.

Each of these subregions is the base of a rectangular box whose heights are given by the $z_i = g(x_i, y_j)$ values. Each box then has a volume of $g(x_i, y_j) dA$. Since dA = (2)(2) = 4, each box has a volume of $g(x_i, y_j) \times 4$.

The approximate values of $g(x_i, y_j)$ are shown below in an array that matches the orientation of the subregions in the previous figure:

37	46	55	60
27	34	42	49
22	27	33	40
16	23	28	34
13	20	25	31
11	18	25	29

The approximate value of $\iint_R^{\square} g(x, y) dA$ is the sum of the volumes of each rectangular box contained within it:

$$\iint_{R}^{\square} g(x, y) \, dA \approx 37(4) + 46(4) + 55(4) + 60(4) + \cdots.$$

Note that the 4 can be factored to the front. Thus, the approximate value of $\iint_R^{\square} g(x, y) dA$ is the sum of all the $g(x_i, y_j)$ values in the array above, multiplied by 4:

$$\begin{split} &\iint_{R}^{\square} g(x,y) \, dA \\ &\approx 4 \begin{pmatrix} 37+46+55+60+27+34+42+49+22+27+33+40 \\ +16+23+28+34+13+20+25+31+11+18+25+29 \end{pmatrix}, \end{split}$$

which is about 2,980 cubic units.

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34. Double Integration over Rectangular Regions

A double integral is evaluated "inside out"—that is, the inside integral is evaluated first, then that result becomes the integrand of the outer integral, which is then evaluated.

Example 34.1: Evaluate $\iint_{R}^{\square} x^2 y \, dA$ where *R* is the rectangle $0 \le x \le 3$ and $1 \le y \le 5$.

Solution: We can choose either the dy dx ordering or the dx dy ordering. Let's choose dA = dx dy. Thus, we have

$$\iint_R^{\square} x^2 y \, dA = \int_1^5 \int_0^3 x^2 y \, dx \, dy.$$

Integrate the inner integral with respect to x, treating y as a constant:

$$\int_0^3 x^2 y \, dx = \left[\frac{1}{3}x^3 y\right]_0^3 = \frac{1}{3}y[3^3 - 0^3] = 9y.$$

Now we integrate the result with respect to *y*:

$$\int_{1}^{5} 9y \, dy = \left[\frac{9}{2}y^{2}\right]_{1}^{5} = \frac{9}{2}(5^{2} - 1^{2}) = 108.$$

If we chose dA = dy dx, we have the following:

$$\int_0^3 \int_1^5 x^2 y \, dy \, dx.$$

The inner integral is determined first with respect to *y*, treating *x* as a constant temporarily:

$$\int_{1}^{5} x^{2} y \, dy = x^{2} \left[\frac{1}{2} y^{2} \right]_{1}^{5} = \frac{1}{2} x^{2} \left[(5)^{2} - (1)^{2} \right] = \frac{1}{2} x^{2} (24) = 12 x^{2}.$$

This result is now integrated with respect to *x*:

$$\int_0^3 12x^2 \ dx = [4x^3]_0^3 = 4[(3)^3 - (0)^3] = 4(27) = 108.$$

Both orderings of the differentials gives the same result, 108, as expected. This is the volume of the solid bounded below by the region of integration *R* and above by the surface $z = x^2y$.

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If the region is infinite in one direction, the integral is improper and may be evaluated using limits.

Example 34.2: Evaluate

$$\int_0^\infty \int_{-1}^2 \frac{x^3}{1+y^2} \, dx \, dy.$$

Solution: The inner integral is determined first, with $\frac{1}{1+y^2}$ moved outside the integral:

$$\int_{-1}^{2} \frac{x^{3}}{1+y^{2}} dx = \frac{1}{1+y^{2}} \int_{-1}^{2} x^{3} dx$$
$$= \frac{1}{1+y^{2}} \left[\frac{1}{4}x^{4}\right]_{-1}^{2}$$
$$= \frac{1}{4} \left(\frac{1}{1+y^{2}}\right) \left[(2)^{4} - (-1)^{4}\right]$$
$$= \frac{15}{4} \left(\frac{1}{1+y^{2}}\right).$$

This is then integrated with respect to y. The constant $\frac{15}{4}$ can be moved outside the integral, and the upper bound, ∞ , is replaced with b, where b is allowed to approach infinity as a limit:

$$\frac{15}{4} \int_0^\infty \frac{1}{1+y^2} \, dy = \lim_{b \to \infty} \left(\frac{15}{4} \int_0^b \frac{1}{1+y^2} \, dy \right)$$
$$= \lim_{b \to \infty} \frac{15}{4} [\arctan y]_0^b$$
$$= \frac{15}{4} \lim_{b \to \infty} [\arctan b - \arctan 0]$$
$$= \frac{15}{4} \left(\frac{\pi}{2} - 0 \right) = \frac{15\pi}{8}.$$

Recall that as an angle θ approaches $\frac{\pi}{2}$ radians from below, $\tan \theta$ approaches positive ∞ . Thus, if $\theta = \arctan b$, then $\arctan b$ approaches $\frac{\pi}{2}$ as *b* approaches ∞ .

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Example 34.3: Evaluate

$$\int_0^4 \int_{\pi/6}^{\pi/2} (x^2 + \cos 3y) \, dy \, dx.$$

Solution: The inner integral is determined first:

$$\int_{\pi/6}^{\pi/2} (x^2 + \cos 3y) \, dy = \left[x^2 y + \frac{1}{3} \sin 3y \right]_{\pi/6}^{\pi/2} \\ = \left[x^2 \left(\frac{\pi}{2} \right) + \frac{1}{3} \sin \left(\frac{3\pi}{2} \right) \right] - \left[x^2 \left(\frac{\pi}{6} \right) + \frac{1}{3} \sin \left(\frac{3\pi}{6} \right) \right]$$

Recall that $\sin\left(\frac{3\pi}{2}\right) = -1$ and that $\sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$. We have,

$$\left[x^{2}\left(\frac{\pi}{2}\right) + \frac{1}{3}(-1)\right] - \left[x^{2}\left(\frac{\pi}{6}\right) + \frac{1}{3}(1)\right] = x^{2}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - \frac{2}{3} = \frac{\pi}{3}x^{2} - \frac{2}{3}$$

This is then integrated:

$$\int_{0}^{4} \left(\frac{\pi}{3}x^{2} - \frac{2}{3}\right) dx = \left[\frac{\pi}{9}x^{3} - \frac{2}{3}x\right]_{0}^{4}$$
$$= \left[\frac{\pi}{9}(4)^{3} - \frac{2}{3}(4)\right] - \left[\frac{\pi}{9}(0)^{3} - \frac{2}{3}(0)\right]$$
$$= \frac{64\pi}{9} - \frac{8}{3}.$$

Example 34.4: Evaluate

$$\int_0^2 \int_1^3 xy e^{x+y^2} dx dy.$$

Solution: We can simplify the integrand using algebra first: $xye^{x+y^2} = xye^xe^{y^2} = xe^xye^{y^2}$. Note that since this is a single term, we may group the factors as desired. The factor xe^x will be integrated using integration by parts, while the factor ye^{y^2} can be integrated using *u*-*du* substitution. It does not make a difference in which order we integrate, but it may be simpler to integrate with respect to y first. Thus, we rewrite the iterated integral as

$$\int_1^3 \int_0^2 x e^x y e^{y^2} \, dy \, dx.$$

Integrating the inside integral with respect to *y*, we have

$$\int_{0}^{2} xe^{x} ye^{y^{2}} dy = xe^{x} \left[\frac{1}{2}e^{y^{2}}\right]_{0}^{2}$$
$$= \frac{1}{2} xe^{x} \left[e^{(2)^{2}} - e^{(0)^{2}}\right]$$
$$= \frac{1}{2} xe^{x} \left[e^{4} - 1\right].$$

This is now integrated with respect to x. Note that $\frac{1}{2}(e^4 - 1)$ is a constant and can be moved outside the integral:

$$\frac{e^4-1}{2}\int_1^3 xe^x\,dx.$$

To antidifferentiate xe^x , use integration by parts. Let u = x and $dv = e^x dx$. Thus, du = dx and $v = e^x$. Since $\int u \, dv = uv - \int v \, du$, we have

$$\frac{e^4 - 1}{2} \int_1^3 x e^x \, dx = \frac{e^4 - 1}{2} \Big[x e^x - \int e^x \, dx \Big]$$
$$= \frac{e^4 - 1}{2} [x e^x - e^x]_1^3$$
$$= \frac{e^4 - 1}{2} [(3e^3 - e^3) - (e^1 - e^1)]$$
$$= \frac{2e^3(e^4 - 1)}{2}$$
$$= e^7 - e^3.$$

Example 34.5: The density of a city's population is given by $P(x, y) = 0.2x^2 + 0.1y^3$, where x and y are in miles, and P is on thousands of people per square mile. Assume that the city is a rectangle measuring 6 miles east to west (x), and 4 miles north to south (y), and that x = 0 and y = 0 is the southwestern corner of the city's boundaries. Find the city's population.

Solution: The city's population is given by the double integral:

$$\int_0^4 \int_0^6 (0.2x^2 + 0.1y^3) \, dx \, dy$$

Evaluating the inside integral with respect to x first, we have

$$\int_{0}^{6} (0.2x^{2} + 0.1y^{3}) dx = \left[\frac{0.2}{3}x^{3} + 0.1xy^{3}\right]_{0}^{6}$$
$$= \left(\frac{0.2}{3}(6)^{3} + 0.1(6)y^{3}\right) - \left(\frac{0.2}{3}(0)^{3} + 0.1(0)y^{3}\right)$$
$$= 14.4 + 0.6y^{3}.$$

This is then integrated with respect to *y*:

$$\int_{0}^{4} (14.4 + 0.6y^{3}) \, dy = \left[14.4y + \frac{0.6}{4} y^{4} \right]_{0}^{4}$$
$$= \left(14.4(4) + \frac{0.6}{4} (4)^{4} \right) - \left(14.4(0) + \frac{0.6}{4} (0)^{4} \right)$$
$$= 96.$$

Thus, the city has about 96,000 people within its boundaries.

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The **average value** of a multivariable function z = f(x, y) over a region *R* is given by $f_{av} = \frac{1}{A(R)} \iint_{R} f(x, y) dA$, where A(R) is the area of region *R*.

Example 34.6: Find the average value of the result in the previous example and explain its meaning in context.

Solution: The region *R* has an area of (6)(4) = 24 square miles. Thus, the average value of $P(x, y) = 0.2x^2 + 0.1y^3$ over *R* is $P_{av} = \frac{1}{24}(96) = 4$. The city has an average density of about 4,000 people per square mile.

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See an error? Have a suggestion? Please see <u>www.surgent.net/vcbook</u>

35. Double Integration over Non-Rectangular Regions of Type I

Consider the region *R* shown below.



The region is bounded by the lines y = 0 (the x-axis), x = 0 (the y-axis), and $y = -\frac{3}{4}x + 3$. If we set up a double integral is the dy dx ordering of integration, we draw an arrow in the positive y direction (see image, above right). It enters the region at $y_1 = 0$ and exits through $y_2 = -\frac{3}{4}x + 3$, where the subscripts help us remember the order in which the boundaries are crossed. The double integral is

$$\int_0^4 \int_{y_1=0}^{y_2=-(3/4)x+3} f(x,y) \, dy \, dx = \int_0^4 \int_0^{-(3/4)x+3} f(x,y) \, dy \, dx$$

As a *dx dy* integral, draw an arrow drawn in the positive *x* direction (see image at right). It enters the region at $x_1 = 0$ and exits through $x_2 = -\frac{4}{3}y + 4$ (which is the equation $y = -\frac{3}{4}x + 3$ that has been solved for *x*). The resulting *y* bounds are 0 to 3, and the double integral is



$$\int_0^3 \int_0^{-(4/3)y+4} f(x,y) \, dx \, dy.$$

There is *no* ambiguity where an arrow enters or exits the region. Such a region is called a **Type I** region. If there is ambiguity, then the region is called a Type II region.

Below, at left, are regions of Type I. At right are regions of Type II.



Integrals over a region of Type I usually require one iterated integral. For regions of Type II, more than one iterated integral is required.

Example 35.1: Find the volume below $z = f(x, y) = xy + x^2$ over the region *R*, which is a triangle with vertices (0,0), (5,0) and (0,10).

Solution. The sketch below shows this to be a region of Type I. Identify all vertex points and the equation of all boundaries.



If we choose to integrate in the dy dx ordering, visualize an arrow drawn in the positive y direction. It enters the region at the x-axis, which is $y_1 = 0$, and exits through $y_2 = -2x + 10$. The x bounds are 0 to 5, and the iterated integral is

$$\int_0^5 \int_0^{-2x+10} (xy+x^2) \, dy \, dx.$$

Integrating with respect to y, we have

$$\int_{0}^{-2x+10} (xy+x^{2}) \, dy = \left[\frac{1}{2}xy^{2}+x^{2}y\right]_{0}^{-2x+10}$$
$$= \left(\frac{1}{2}x(-2x+10)^{2}+x^{2}(-2x+10)\right) - \left(\frac{1}{2}x(0)^{2}+x^{2}(0)\right).$$

The expression above simplifies to $-10x^2 + 50x$. This is the integrand to be integrated with respect to x now:

$$\int_0^5 (-10x^2 + 50x) \, dx = \left[-\frac{10}{3}x^3 + 25x^2 \right]_0^5$$
$$= \left(-\frac{10}{3}(5)^3 + 25(5)^2 \right) - 0$$
$$= \frac{625}{3}.$$

Example 35.2: Evaluate

$$\iint_R^{\square} 2xy^2 \, dA,$$

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where *R* is in the first quadrant bounded by the *x*-axis, the *y*-axis and the parabola $y = 25 - x^2$.

Solution: Sketch the region and decide on an ordering of integration. If we choose a *dy dx* ordering, visualize an arrow drawn in the positive *y* direction. It enters the region at the *x*-axis, which is $y_1 = 0$, and exits through the parabola $y_2 = 25 - x^2$. The bounds for *x* are 0 to 5.



The double integral is

$$\int_0^5 \int_0^{25-x^2} 2xy^2 \, dy \, dx.$$

The inside integral is determined:

$$\int_0^{25-x^2} 2xy^2 \, dy = \left[\frac{2}{3}xy^3\right]_0^{25-x^2} = \frac{2}{3}x(25-x^2)^3.$$

This is integrated with respect to *x* using a *u*-*du* substitution, with $u = 25 - x^2$:

$$\int_{0}^{5} \frac{2}{3} x (25 - x^{2})^{3} dx = \left[-\frac{1}{12} (25 - x^{2})^{4} \right]_{0}^{5}$$
$$= \left(-\frac{1}{12} (25 - (5)^{2})^{4} \right) - \left(-\frac{1}{12} (25 - (0)^{2})^{4} \right)$$
$$= 0 - \left(-\frac{1}{12} (25)^{4} \right) = \frac{390,625}{12}.$$

If we use a dx dy ordering, the double integral is written

$$\int_0^{25} \int_0^{\sqrt{25-y}} 2xy^2 \, dx \, dy,$$





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Example 35.3: Given

$$\int_0^5 \int_{e^x}^{e^5} g(x,y) \, dy \, dx,$$

Reverse the order of integration (that is, rewrite this double integral as a dx dy integral).

Solution: The ordering of integration tells us that if we visualize an arrow in the positive y direction, it will enter the region at $y_1 = e^x$ and exit at the line $y = e^5$, with the x bounds being 0 to 5. The region is shown below, with all vertices and boundaries identified:



To reverse the ordering, now visualize an arrow in the positive x direction. It enters at $x_1 = 0$ (the y-axis) and exits at $x_2 = \ln y$. The bounds for y are 1 to e^5 . We have

$$\int_1^{e^5}\int_0^{\ln y} g(x,y)\,dx\,dy.$$

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Example 35.4: Reverse the order of integration of

$$\int_0^9 \int_{y/3}^{\sqrt{y}} h(x,y) \, dx \, dy.$$

Solution: Visualize an arrow in the positive *x* direction. It enters the region *R* at $x_1 = \frac{1}{3}y$ and exits at $x_2 = \sqrt{y}$. The two graphs meet at (0,0) and (3,9), and the region is shown below:



Observe that we redefined the bounds in y as a function in terms of x.

Viewing the region, now visualize an arrow in the positive y direction. It will enter R at $y_1 = x^2$ and exit at $y_2 = 3x$. These become the bounds for the dy integral. The bounds for x are 0 to 3, and the equivalent double integral in the dy dx ordering is

$$\int_0^3 \int_{x^2}^{3x} h(x, y) \, dy \, dx.$$

Example 35.5: Evaluate

$$\int_0^2 \int_y^2 \sqrt{1+x^2} \, dx \, dy.$$

Solution: If we attempt to evaluate the integrals as written (inside first with respect to *x*, then outside with respect to *y*), we discover that finding the antiderivative of $\sqrt{1 + x^2}$ with respect to *x* is challenging (it would require a trigonometric substitution). Instead, we reverse the order of integration.

The double integral, as written, suggests that the region *R* is bounded by the line x = y and the line x = 2, with the bounds for *y* being 0 to 2. This region is sketched below, and all vertices and boundaries are identified:



Reversing the order of integration, we visualize an arrow in the positive ydirection. It enters R at $y_1 = 0$ and exits at $y_2 = x$. The bounds for x will be 0 to 2, and the double integral in the dy dx ordering is

$$\int_{0}^{2} \int_{0}^{x} \sqrt{1+x^{2}} \, dy \, dx.$$

Now, the inside integral is determined. Note that the antiderivative of $\sqrt{1 + x^2}$ with respect to *y* is $y\sqrt{1 + x^2}$. Thus, we have

$$\int_0^x \sqrt{1+x^2} \, dy = \left[y\sqrt{1+x^2} \right]_0^x = x\sqrt{1+x^2}.$$

Now we integrate $x\sqrt{1+x^2}$ with respect to *x*. The antiderivative of $x\sqrt{1+x^2}$ is found by a *u*-*du* substitution. We have

$$\int_0^2 x\sqrt{1+x^2} \, dx = \left[\frac{1}{3}(1+x^2)^{3/2}\right]_0^2 = \frac{1}{3}(5^{3/2}-1).$$