# 33. Riemann Summation over Rectangular Regions

A rectangular region *R* in the *xy*-plane can be defined using compound inequalities, where *x* and *y* are each bound by constants such that $a_1 \le x \le a_2$ and $b_1 \le y \le b_2$. Let $z = f(x, y)$ be a continuous function defined over a rectangular region *R* in the *xy*-plane. The notation

$$\iint_R f(x,y)\,dA$$

represents the **double integral** of $z = f(x, y)$ over *R*. The $dA$ represents "area element", and is either $dy\,dx$ or $dx\,dy$. Thus, we can write

$$\iint_R f(x,y)\,dA = \int_{a_1}^{a_2}\int_{b_1}^{b_2} f(x,y)\;dy\;dx = \int_{b_1}^{b_2}\int_{a_1}^{a_2} f(x,y)\;dx\;dy.$$

Note that the bounds $a_1$ and $a_2$ agree with the differential *dx*, and bounds $b_1$ and $b_2$ agree with *dy*.

The value of a double integral can be approximated by **Riemann sums** adapted to the two-dimensional case. Interval $a_1 \le x \le a_2$ is subdivided into *m* subdivisions (not necessarily of equal size) and interval $b_1 \le y \le b_2$ is subdivided into *n* subdivisions (again, not necessarily of equal size). If we define indices $1 \le i \le m$ and $1 \le j \le n$, then we have a way to identify a particular subdivision within region *R*. For example, if $a_1 \le x \le a_2$ is subdivided into 4 subdivisions and $b_1 \le y \le b_2$ is subdivided into 5 subdivisions, then $(x_2, y_3)$ is a representative point within the 2nd subdivision of the *x*-interval and the 3rd subdivision of the *y*-interval, and $f(x_2, y_3)$ is the function evaluated at $(x_2, y_3)$.

Using this scheme, a double integral can be approximated by a double sum over $i$ and $j$:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_m, x_n)\,\Delta y\,\Delta x \;\text{ or }\; \sum_{j=1}^{n}\sum_{i=1}^{m} f(x_m, x_n)\,\Delta x\,\Delta y.$$

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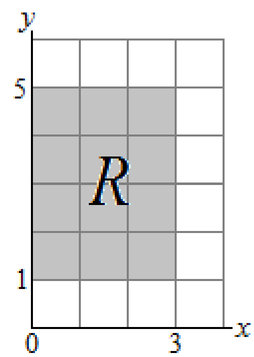
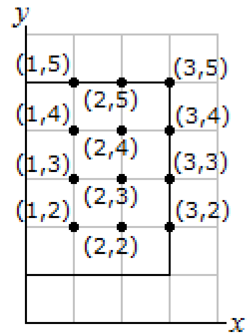
**Example 33.1:** Use Riemann Sums to approximate the value of $\iint_R x^2 y\,dA$ where *R* is the rectangle $0 \le x \le 3$ and $1 \le y \le 5$ in the *xy* plane. Subdivide the region *R* into subregions each with length 1 to a side, and from each subregion, choose *x* and *y* to be the "upper right" corner.




**Solution:** The rectangular region *R* is shown at right, subdivided into subregions, so that $\Delta A = \Delta x\,\Delta y = (1)(1) = 1$. There are 12 such subregions.

Then choose a representative point $(x_i, y_j)$ within each subregion. In this example, we choose $(x_i, y_j)$ to be the "upper right" point within each subregion (this is an arbitrary choice. We could choose the "lower left" or the "middle point", and so on). Here, $1 \le i \le 3$ and $2 \le j \le 5$, the bounds chosen for convenience.




Next, evaluate the integrand $z = f(x, y) = x^2y$ at the representative points $(x_i, y_j)$:

$$
\begin{array}{lll}
f(1,5) = 5 & f(2,5) = 20 & f(3,5) = 45 \\
f(1,4) = 4 & f(2,4) = 16 & f(3,4) = 36 \\
f(1,3) = 3 & f(2,3) = 12 & f(3,3) = 27 \\
f(1,2) = 2 & f(2,2) = 8 & f(3,2) = 18
\end{array}
$$

Visually, we have a surface $z = f(x, y) = x^2y$ "above" the *xy*-plane. Each subregion in *R* is the base of a rectangular box whose height is the function value shown in the table above. Each box has a volume of $f(x_i, y_j)\, dA$. Since $dA = dx\, dy = (1)(1) = 1$ in each case, each box has volume $f(x_i, y_j) \times 1$, or simply $f(x_i, y_j)$. The value of $\iint_R x^2y\, dA$ is approximated by the sum of the volumes of the rectangular boxes contained within it. Thus,

$$
\begin{aligned}
\iint_R x^2y\, dA &\approx \sum_{i=1}^{3} \sum_{j=2}^{5} f(x_m, x_n)\, \Delta y\, \Delta x \\
&= 2 + 8 + 18 + 3 + 12 + 27 + 4 + 16 + 36 + 5 + 20 + 45 \\
&= 196.
\end{aligned}
$$

Note that if we chose the representative point to be the lower-left corner of each subregion, we would find that $\iint_R x^2y\, dA \approx 50$. The mean, $\frac{196+50}{2} = 123$, is a reasonable approximation of $\iint_R x^2y\, dA$.

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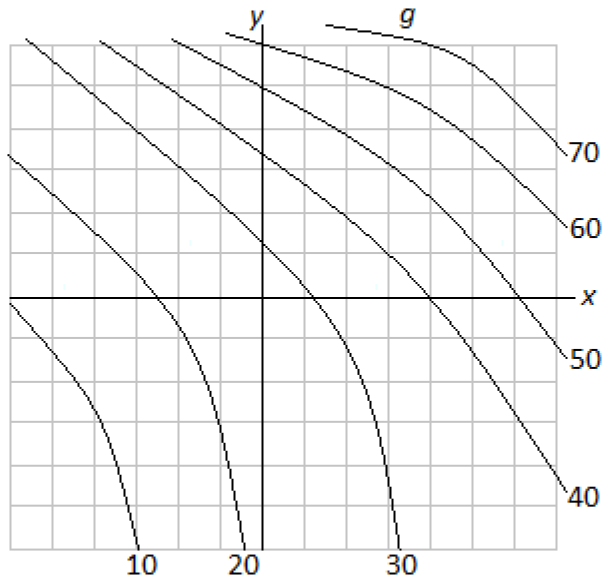
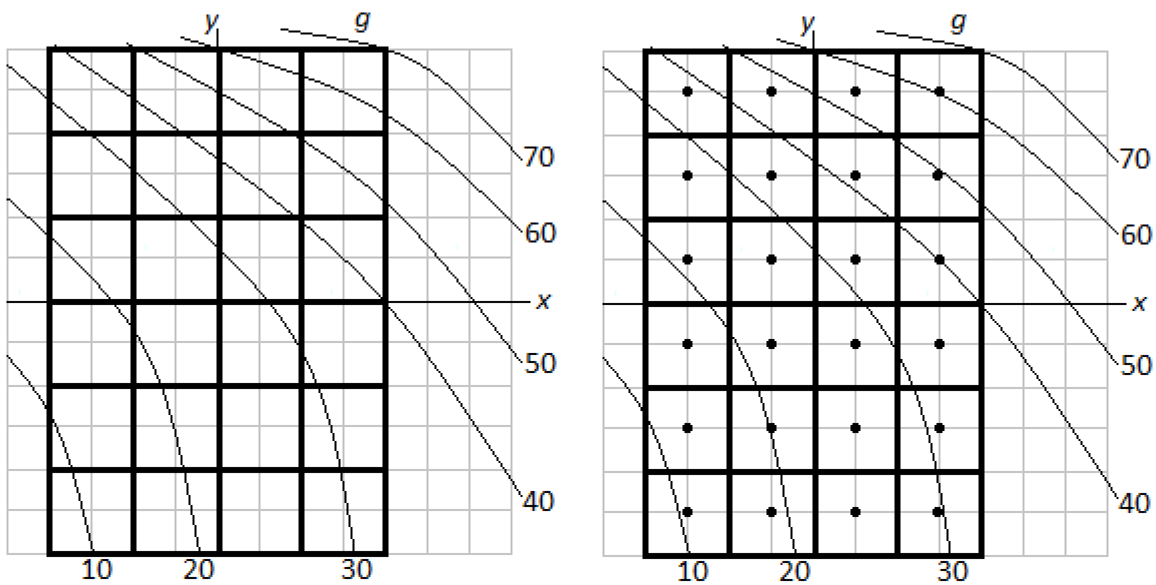
The numbering of the subscripts $i$ and $j$ is often adapted to each problem and made as convenient as possible. It is simply a way to track each subdivision within the region *R*. In the previous example, had we used "lower left" corners as the representative point of each subregion $\Delta A$, we could have defined $0 \le i \le 2$ and $1 \le j \le 4$.

**Example 33.2:** Use Riemann Sums to approximate $\iint_R g(x,y)\,dA$, where $g$ is shown by the contour map below. Let the region of integration $R$ be given by $-4 \le x \le 4, -6 \le y \le 6$, and let $\Delta x = 2$ and $\Delta y = 2$. Use the middle point within each subregion.




**Solution:** The region $R$ is identified and then subdivided into $2 \times 2$ subregions (lower left, boldfaced). Then the middle point $(x_i, y_j)$ from within each subregion is identified (lower right):




The values of $z = g(x,y)$ are estimated from the contour map. For example, in the top tier of subregions, reading left to right and using the middle points, the values of $g$ are approximately $g(-3,5) = 37, g(-1,5) = 46, g(1,5) = 55$ and $g(3,5) = 60$.

Each of these subregions is the base of a rectangular box whose heights are given by the $z_i = g(x_i, y_j)$ values. Each box then has a volume of $g\left(x_i, y_j\right)\, dA$. Since $dA = (2)(2) = 4$, each box has a volume of $g\left(x_i, y_j\right) \times 4$.

The approximate values of $g(x_i, y_j)$ are shown below in an array that matches the orientation of the subregions in the previous figure:

| | | | |
|---|---|---|---|
| 37 | 46 | 55 | 60 |
| 27 | 34 | 42 | 49 |
| 22 | 27 | 33 | 40 |
| 16 | 23 | 28 | 34 |
| 13 | 20 | 25 | 31 |
| 11 | 18 | 25 | 29 |

The approximate value of $\iint_R\ g(x, y)\, dA$ is the sum of the volumes of each rectangular box contained within it:

$$\iint_R\ g(x, y)\, dA \approx 37(4) + 46(4) + 55(4) + 60(4) + \cdots.$$

Note that the 4 can be factored to the front. Thus, the approximate value of $\iint_R\ g(x, y)\, dA$ is the sum of all the $g(x_i, y_j)$ values in the array above, multiplied by 4:

$$\iint_R\ g(x, y)\, dA \approx 4\begin{pmatrix} 37 + 46 + 55 + 60 + 27 + 34 + 42 + 49 + 22 + 27 + 33 + 40 + \\ 16 + 23 + 28 + 34 + 13 + 20 + 25 + 31 + 11 + 18 + 25 + 29 \end{pmatrix},$$

which is about 2,980 cubic units.

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# 34. Double Integration over Rectangular Regions

A double integral is evaluated "inside out"—that is, the inside integral is evaluated first, then that result becomes the integrand of the outer integral, which is then evaluated.

**Example 34.1:** Evaluate $\iint_R \; x^2y\,dA$ where *R* is the rectangle $0 \le x \le 3$ and $1 \le y \le 5$.

**Solution:** We can choose either the $dy\,dx$ ordering or the $dx\,dy$ ordering. Let's choose $dA = dx\,dy$. Thus, we have

$$\iint_R \; x^2y\,dA = \int_1^5 \int_0^3 x^2y\,dx\,dy.$$

Integrate the inner integral with respect to *x*, treating *y* as a constant:

$$\int_0^3 x^2y\,dx = \left[\frac{1}{3}x^3y\right]_0^3 = \frac{1}{3}y[3^3 - 0^3] = 9y.$$

Now we integrate the result with respect to *y*:

$$\int_1^5 9y\,dy = \left[\frac{9}{2}y^2\right]_1^5 = \frac{9}{2}(5^2 - 1^2) = 108.$$

If we chose $dA = dy\,dx$, we have the following:

$$\int_0^3 \int_1^5 x^2y\,dy\,dx.$$

The inner integral is determined first with respect to *y*, treating *x* as a constant temporarily:

$$\int_1^5 x^2y\,dy = x^2\left[\frac{1}{2}y^2\right]_1^5 = \frac{1}{2}x^2[(5)^2 - (1)^2] = \frac{1}{2}x^2(24) = 12x^2.$$

This result is now integrated with respect to *x*:

$$\int_0^3 12x^2\;dx = [4x^3]_0^3 = 4[(3)^3 - (0)^3] = 4(27) = 108.$$

Both orderings of the differentials gives the same result, 108, as expected. This is the volume of the solid bounded below by the region of integration *R* and above by the surface $z = x^2y$.

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If the region is infinite in one direction, the integral is improper and may be evaluated using limits.

**Example 34.2:** Evaluate

$$\int_0^{\infty}\int_{-1}^{2}\frac{x^3}{1+y^2}\;dx\;dy.$$

**Solution:** The inner integral is determined first, with $\frac{1}{1+y^2}$ moved outside the integral:

$$\begin{aligned}\int_{-1}^{2}\frac{x^3}{1+y^2}\;dx &= \frac{1}{1+y^2}\int_{-1}^{2}x^3\;dx\\ &= \frac{1}{1+y^2}\left[\frac{1}{4}x^4\right]_{-1}^{2}\\ &= \frac{1}{4}\left(\frac{1}{1+y^2}\right)[(2)^4-(-1)^4]\\ &= \frac{15}{4}\left(\frac{1}{1+y^2}\right).\end{aligned}$$

This is then integrated with respect to $y$. The constant $\frac{15}{4}$ can be moved outside the integral, and the upper bound, ∞, is replaced with $b$, where $b$ is allowed to approach infinity as a limit:

$$\begin{aligned}\frac{15}{4}\int_0^{\infty}\frac{1}{1+y^2}\;dy &= \lim_{b\to\infty}\left(\frac{15}{4}\int_0^{b}\frac{1}{1+y^2}\;dy\right)\\ &= \lim_{b\to\infty}\frac{15}{4}[\arctan y]_0^b\\ &= \frac{15}{4}\lim_{b\to\infty}[\arctan b-\arctan 0]\\ &= \frac{15}{4}\left(\frac{\pi}{2}-0\right)=\frac{15\pi}{8}.\end{aligned}$$

Recall that as an angle $\theta$ approaches $\frac{\pi}{2}$ radians from below, $\tan\theta$ approaches positive ∞. Thus, if $\theta=\arctan b$, then $\arctan b$ approaches $\frac{\pi}{2}$ as $b$ approaches ∞.

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**Example 34.3:** Evaluate

$$\int_0^4 \int_{\pi/6}^{\pi/2} (x^2 + \cos 3y)\, dy\ dx.$$

**Solution:** The inner integral is determined first:

$$\int_{\pi/6}^{\pi/2} (x^2 + \cos 3y)\, dy = \left[x^2 y + \frac{1}{3}\sin 3y\right]_{\pi/6}^{\pi/2}$$
$$= \left[x^2\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right)\right] - \left[x^2\left(\frac{\pi}{6}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{6}\right)\right].$$

Recall that $\sin\left(\frac{3\pi}{2}\right) = -1$ and that $\sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$. Thus we have,

$$\left[x^2\left(\frac{\pi}{2}\right) + \frac{1}{3}(-1)\right] - \left[x^2\left(\frac{\pi}{6}\right) + \frac{1}{3}(1)\right] = x^2\left(\frac{\pi}{2} - \frac{\pi}{6}\right) - \frac{2}{3} = \frac{\pi}{3}x^2 - \frac{2}{3}.$$

This is then integrated:

$$\int_0^4 \left(\frac{\pi}{3}x^2 - \frac{2}{3}\right)\ dx = \left[\frac{\pi}{9}x^3 - \frac{2}{3}x\right]_0^4$$
$$= \left[\frac{\pi}{9}(4)^3 - \frac{2}{3}(4)\right] - \left[\frac{\pi}{9}(0)^3 - \frac{2}{3}(0)\right]$$
$$= \frac{64\pi}{9} - \frac{8}{3}.$$

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**Example 34.4:** Evaluate

$$\int_0^2 \int_1^3 xye^{x+y^2}\ dx\ dy.$$

**Solution:** We can simplify the integrand using algebra first: $xye^{x+y^2} = xye^x e^{y^2} = xe^x ye^{y^2}$. Note that since this is a single term, we may group the factors as desired. The factor $xe^x$ will be integrated using integration by parts, while the factor $ye^{y^2}$ can be integrated using *u*-*du* substitution. It does not make a difference in which order we integrate, but it may be simpler to integrate with respect to *y* first. Thus, we rewrite the iterated integral as

$$\int_1^3 \int_0^2 xe^x ye^{y^2}\ dy\, dx.$$

Integrating the inside integral with respect to $y$, we have

$$\int_0^2 xe^x ye^{y^2}\, dy = xe^x\left[\frac{1}{2}e^{y^2}\right]_0^2$$
$$= \frac{1}{2}xe^x\left[e^{(2)^2} - e^{(0)^2}\right]$$
$$= \frac{1}{2}xe^x[e^4 - 1].$$

This is now integrated with respect to $x$. Note that $\frac{1}{2}(e^4 - 1)$ is a constant and can be moved outside the integral:

$$\frac{e^4 - 1}{2}\int_1^3 xe^x\, dx.$$

To antidifferentiate $xe^x$, use integration by parts. Let $u = x$ and $dv = e^x dx$. Thus, $du = dx$ and $v = e^x$. Since $\int u\, dv = uv - \int v\, du$, we have

$$\frac{e^4 - 1}{2}\int_1^3 xe^x\, dx = \frac{e^4 - 1}{2}\left[xe^x - \int e^x dx\right]$$
$$= \frac{e^4 - 1}{2}[xe^x - e^x]_1^3$$
$$= \frac{e^4 - 1}{2}[(3e^3 - e^3) - (e^1 - e^1)]$$
$$= \frac{2e^3(e^4 - 1)}{2}$$
$$= e^7 - e^3.$$

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**Example 34.5:** The density of a city's population is given by $P(x, y) = 0.2x^2 + 0.1y^3$, where $x$ and $y$ are in miles, and $P$ is on thousands of people per square mile. Assume that the city is a rectangle measuring 6 miles east to west ($x$), and 4 miles north to south ($y$), and that $x = 0$ and $y = 0$ is the southwestern corner of the city's boundaries. Find the city's population.

**Solution:** The city's population is given by the double integral:

$$\int_0^4\int_0^6 (0.2x^2 + 0.1y^3)\, dx\, dy.$$

Evaluating the inside integral with respect to *x* first, we have

$$\int_0^6 (0.2x^2 + 0.1y^3)\,dx = \left[\frac{0.2}{3}x^3 + 0.1xy^3\right]_0^6$$
$$= \left(\frac{0.2}{3}(6)^3 + 0.1(6)y^3\right) - \left(\frac{0.2}{3}(0)^3 + 0.1(0)y^3\right)$$
$$= 14.4 + 0.6y^3.$$

This is then integrated with respect to *y*:

$$\int_0^4 (14.4 + 0.6y^3)\,dy = \left[14.4y + \frac{0.6}{4}y^4\right]_0^4$$
$$= \left(14.4(4) + \frac{0.6}{4}(4)^4\right) - \left(14.4(0) + \frac{0.6}{4}(0)^4\right)$$
$$= 96.$$

Thus, the city has about 96,000 people within its boundaries.

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The **average value** of a multivariable function $z = f(x, y)$ over a region $R$ is given by $f_{av} = \frac{1}{A(R)} \iint_R f(x, y)\,dA$, where $A(R)$ represents the area of region $R$.

**Example 34.6:** Find the average value of the result in the previous example, and explain its meaning in context.

**Solution:** The region $R$ has an area of $(6)(4) = 24$ square miles. Thus, the average value of $P(x, y) = 0.2x^2 + 0.1y^3$ over $R$ is $P_{av} = \frac{1}{24}(96) = 4$. The city has an average density of about 4,000 people per square mile.

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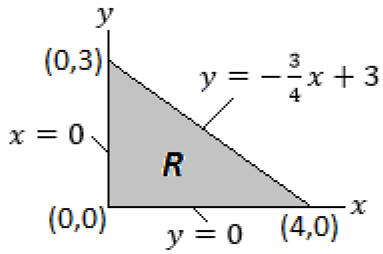
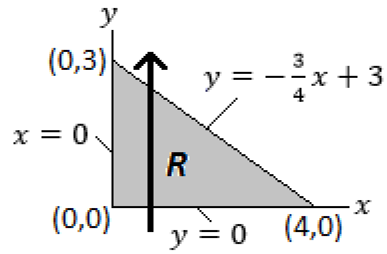
# 35. Double Integration over Non-Rectangular Regions of Type I

Consider the region $R$ shown below.

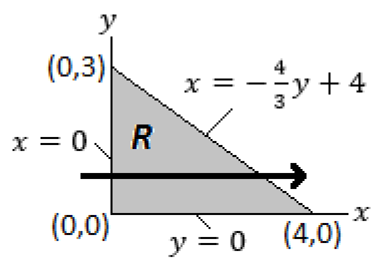



The region is bounded by the lines $y = 0$ (the $x$-axis), $x = 0$ (the $y$-axis), and $y = -\frac{3}{4}x + 3$. If we set up a double integral is the $dy\ dx$ ordering of integration, we draw an arrow in the positive $y$ direction (see image, above right). It enters the region at $y_1 = 0$ and exits through $y_2 = -\frac{3}{4}x + 3$, where the subscripts help us remember the order in which the boundaries are crossed. The double integral is

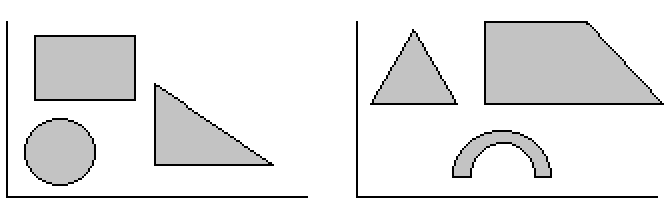
$$\int_0^4 \int_{y_1=0}^{y_2=-(3/4)x+3} f(x,y)\ dy\ \ dx = \int_0^4 \int_0^{-(3/4)x+3} f(x,y)\ dy\ \ dx.$$

As a $dx\ dy$ integral, draw an arrow drawn in the positive $x$ direction (see image at right). It enters the region at $x_1 = 0$ and exits through $x_2 = -\frac{4}{3}y + 4$ (which is the equation $y = -\frac{3}{4}x + 3$ that has been solved for $x$). The resulting $y$ bounds are 0 to 3, and the double integral is




$$\int_0^3 \int_0^{-(4/3)x+4} f(x,y)\ dx\ \ dy.$$

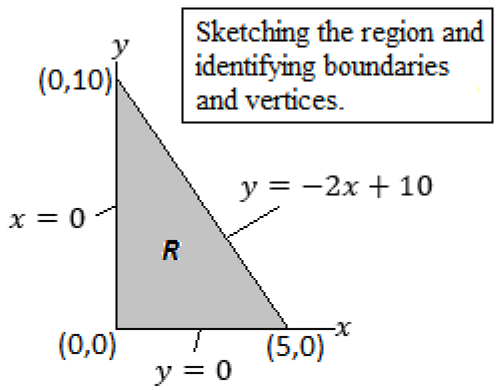
There is *no* ambiguity where an arrow enters or exits the region. Such a region is called a **Type I** region. If there is ambiguity, then the region is called a Type II region. Below, at left, are regions of Type I. At right are regions of Type II.

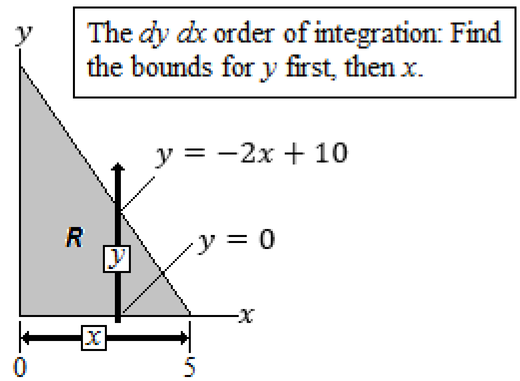


Integrals over a region of Type I usually require one iterated integral. For regions of Type II, more than one iterated integral is required.

**Example 35.1:** Find the volume below $z = f(x,y) = xy + x^2$ over the region $R$, which is a triangle with vertices (0,0), (5,0) and (0,10).

**Solution.** The sketch below shows this to be a region of Type I. Identify all vertex points and the equation of all boundaries.




If we choose to integrate in the $dy\,dx$ ordering, visualize an arrow drawn in the positive $y$ direction. It enters the region at the $x$-axis, which is $y_1 = 0$, and exits through $y_2 = -2x + 10$. The $x$ bounds are 0 to 5, and the iterated integral is

$$\int_0^5 \int_0^{-2x+10} (xy + x^2)\ dy\,dx.$$

Integrating with respect to $y$, we have

$$\int_0^{-2x+10} (xy + x^2)\ dy = \left[\frac{1}{2}xy^2 + x^2y\right]_0^{-2x+10}$$
$$= \left(\frac{1}{2}x(-2x+10)^2 + x^2(-2x+10)\right) - \left(\frac{1}{2}x(0)^2 + x^2(0)\right).$$

The expression above simplifies to $-10x^2 + 50x$. This is the integrand to be integrated with respect to $x$ now:

$$\int_0^5 (-10x^2 + 50x)\,dx = \left[-\frac{10}{3}x^3 + 25x^2\right]_0^5$$
$$= \left(-\frac{10}{3}(5)^3 + 25(5)^2\right) - 0$$
$$= \frac{625}{3}.$$

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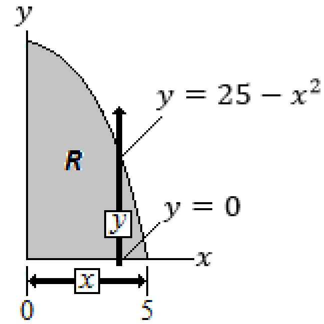
**Example 35.2:** Evaluate

$$\iint_R 2xy^2\,dA,$$

where $R$ is in the first quadrant bounded by the $x$-axis, the $y$-axis and the parabola $y = 25 - x^2$.

**Solution:** Sketch the region and decide on an ordering of integration. The region is shown to the right. If we choose a $dy\,dx$ ordering, visualize an arrow drawn in the positive $y$ direction. It enters the region at the $x$-axis, which is $y_1 = 0$, and exits through $y_2 = 25 - x^2$. The bounds for $x$ are 0 to 5, and the double integral is




The $dy\,dx$ order of integration. Find the bounds for $y$ first, then $x$.

$$\int_0^5 \int_0^{25-x^2} 2xy^2\;dy\,dx.$$

The inside integral is determined:

$$\int_0^{25-x^2} 2xy^2\;dy = \left[\frac{2}{3}xy^3\right]_0^{25-x^2} = \frac{2}{3}x(25-x^2)^3.$$

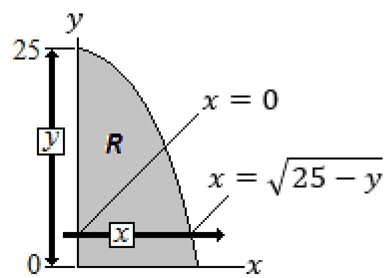
This is then integrated with respect to $x$ using a $u$-$du$ substitution, with $u = 25 - x^2$:

$$\int_0^5 \frac{2}{3}x(25-x^2)^3\,dx = \left[-\frac{1}{12}(25-x^2)^4\right]_0^5$$

$$= \left(-\frac{1}{12}(25-(5)^2)^4\right) - \left(-\frac{1}{12}(25-(0)^2)^4\right)$$

$$= 0 - \left(-\frac{1}{12}(25)^4\right) = \frac{390{,}625}{12}.$$

If we use a $dx\,dy$ ordering, the double integral is written




The $dx\,dy$ order of integration. Find the bounds for $x$ first, then $y$.

$$\int_0^{25} \int_0^{\sqrt{25-y}} 2xy^2\;dx\,dy.$$

It also evaluates to $\frac{390{,}625}{12}$.

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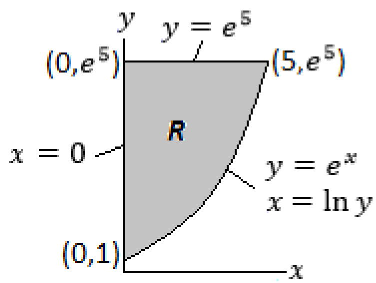
**Example 35.3:** Given

$$\int_0^5 \int_{e^x}^{e^5} g(x,y)\, dy\, dx,$$

Reverse the order of integration (that is, rewrite this double integral as a $dx\, dy$ integral).

**Solution:** The ordering of integration tells us that if we visualize an arrow in the positive *y* direction, it will enter the region at $y_1 = e^x$ and exit at the line $y = e^5$, with the *x* bounds being 0 to 5. The region is shown below, with all vertices and boundaries identified:




To reverse the ordering, now visualize an arrow in the positive *x* direction. It enters at $x_1 = 0$ (the $y$-axis) and exits at $x_2 = \ln y$. The bounds for *y* are 1 to $e^5$. We have

$$\int_1^{e^5} \int_0^{\ln y} g(x,y)\, dx\, dy.$$

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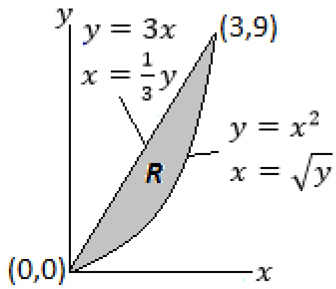
**Example 35.4:** Reverse the order of integration of

$$\int_0^9 \int_{y/3}^{\sqrt{y}} h(x,y)\,dx\,dy.$$

**Solution:** Visualize an arrow in the positive *x* direction. It enters the region $R$ at $x_1 = \frac{1}{3}y$ and exits at $x_2 = \sqrt{y}$. The two graphs meet at (0,0) and (3,9), and the region is shown below:




Observe that we redefined the bounds in *y* as a function in terms of *x*.

Viewing the region, now visualize an arrow in the positive *y* direction. It will enter $R$ at $y_1 = x^2$ and exit at $y_2 = 3x$. These become the bounds for the *dy* integral. The bounds for *x* are 0 to 3, and the equivalent double integral in the $dy\,dx$ ordering is

$$\int_0^3 \int_{x^2}^{3x} h(x,y)\,dy\,dx.$$

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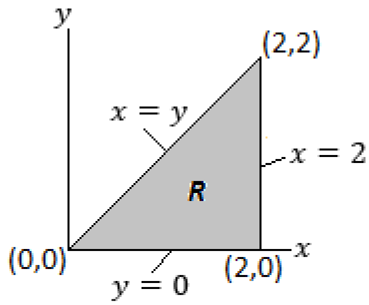
**Example 35.5:** Evaluate

$$\int_0^2 \int_y^2 \sqrt{1+x^2}\, dx\ dy.$$

**Solution:** If we attempt to evaluate the integrals as written (inside first with respect to *x*, then outside with respect to *y*), we discover that finding the antiderivative of $\sqrt{1+x^2}$ with respect to *x* is challenging (it would require a trigonometric substitution). Instead, we reverse the order of integration.

The double integral, as written, suggests that the region *R* is bounded by the line $x = y$ and the line $x = 2$, with the bounds for *y* being 0 to 2. This region is sketched below, and all vertices and boundaries are identified:




Reversing the order of integration, we visualize an arrow in the positive *y*-direction. It enters *R* at $y_1 = 0$ and exits at $y_2 = x$. The bounds for *x* will be 0 to 2, and the double integral in the $dy\ dx$ ordering is

$$\int_0^2 \int_0^x \sqrt{1+x^2}\ dy\ dx.$$

Now, the inside integral is determined. Note that the antiderivative of $\sqrt{1+x^2}$ with respect to *y* is $y\sqrt{1+x^2}$. Thus, we have

$$\int_0^x \sqrt{1+x^2}\ dy = \left[y\sqrt{1+x^2}\right]_0^x = x\sqrt{1+x^2}.$$

Now we integrate $x\sqrt{1+x^2}$ with respect to *x*. The antiderivative of $x\sqrt{1+x^2}$ is found by a *u*-*du* substitution. We have

$$\int_0^2 x\sqrt{1+x^2}\ dx = \left[\frac{1}{3}(1+x^2)^{3/2}\right]_0^2 = \frac{1}{3}\left(5^{3/2} - 1\right).$$

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