22. Projectile Motion

On Earth, the gravitational acceleration constant is -9.8 meters per second per second (or $\mathrm{m/s^2}$). If we superimpose an *xy*-axis system with the positive *y* axis being "up", then **acceleration** can be written as a vector-valued function,

$$\mathbf{a}(t) = \langle 0, -9.8 \rangle.$$

The *x*-component of acceleration is 0, since falling bodies will not accelerate horizontally due to gravity. Note that both components are constants.

Integrating acceleration, we get **velocity**, which is also a vector-valued function:

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\ dt \\ &= \int \langle 0, -9.8 \rangle\ dt \\ &= \langle 0, -9.8t \rangle + \langle v_x, v_y \rangle \\ &= \langle v_x, -9.8t + v_y \rangle.\end{aligned}$$

Here, $\langle v_x, v_y \rangle$ are constants of integration and represents the initial velocity in the *x*-direction and in the *y*-direction, respectively. **Speed** is the magnitude of velocity. Speed is a scalar value.

Integrating velocity, we get **displacement** (or **position**), also a vector-valued function:

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\ dt \\ &= \int \langle v_x, -9.8t + v_y \rangle\ dt \\ &= \langle v_x t + r_x, -4.9t^2 + v_y t + r_y \rangle.\end{aligned}$$

Similar to above, $\langle r_x, r_y \rangle$ represent the initial position of the object in the *x*-direction and *y*-direction respectively. The placement of the origin is arbitrary but is usually done so that $r_x = 0$ since it is almost always practical to assume no initial horizontal distance.

A typical object that is released and then allowed to return to earth under gravity alone will follow a parabolic arc. Its position vector $\mathbf{r}(t)$, velocity vector $\mathbf{v}(t)$ and acceleration vector $\mathbf{a}(t)$ are a "tool kit" of equations that completely describes the motion of the object. In most projectile motion situations, we assume that air friction is ignored, any spin on the object is not considered, and that time t is in seconds and is usually reckoned from $t = 0$, representing when the object was released.

Here are some common questions that may be posed regarding a falling object:

• What is the maximum height of the object? When does it achieve its maximum height?
• How far away does the object travel in the horizontal direction? (Its **range**).
• How fast is the object travelling when it impacts the ground for the first time?

Build the complete set of displacement, velocity and acceleration vectors first, before solving any subsequent questions.

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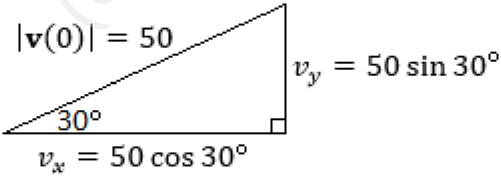
Example 22.1: A ball is propelled off a 100 m tall cliff at an initial speed of 50 meters per second at an angle of 30 degrees above the horizontal.

a) Find the maximum height of the ball.
b) Find the range of the ball when hits the ground for the first time.
c) How fast is the ball travelling when it impacts the ground for the first time?
d) At what angle does the ball impact the ground for the first time?

Solution: Starting with acceleration, $\mathbf{a}(t) = \langle 0, -9.8 \rangle$, integrate to obtain

$$\mathbf{v}(t) = \langle v_x, -9.8t + v_y \rangle.$$

To find v_x and v_y, note that its initial speed is $|\mathbf{v}(0)| = 50$ at an angle of 30 degrees. This suggests a right triangle, in which $|\mathbf{v}(0)| = 50$ is the hypotenuse, and v_x and v_y are the horizontal and vertical legs, respectively.



Thus, $v_x = 50 \cos 30 = 25\sqrt{3} \approx 43.301$, and $v_y = 50 \sin 30 = 25$, and the velocity vector is

$$\mathbf{v}(t) = \langle 43.301, -9.8t + 25 \rangle.$$

Integrating $\mathbf{v}$, we obtain the displacement vector

$$\mathbf{r}(t) = \langle 43.301t + r_x, -4.9t^2 + 25t + r_y \rangle.$$

Since the ball was thrown off the top of a cliff, set the origin at the base of the cliff, so that $r_x = 0$ and $r_y = 100$. Therefore, the displacement vector is

$$\mathbf{r}(t) = \langle 43.301t, -4.9t^2 + 25t + 100\rangle.$$

We now have sufficient information to answer the posed questions. In all cases, assume $t \geq 0$.

a) The ball reaches its maximum height when the *y*-component of velocity is 0 since the ball's vertical velocity is 0 (momentarily stops) when it reaches it maximum height. Thus, we solve $-9.8t + 25 = 0$ to find *when* the ball has reached its maximum height. This happens at $t = 25/9.8 = 2.551$ seconds. Substituting this into the *y*-component of displacement, we now determine the height of the ball at this time:

$$\text{Maximum height} = -4.9(2.551)^2 + 25(2.551) + 100 = 131.888 \text{ meters.}$$

b) The ball impacts the ground when the *y*-component of displacement is 0. We use the quadratic formula to find the roots of $-4.9t^2 + 25t + 100 = 0$:

$$t = \frac{-25 \pm \sqrt{25^2 - 4(-4.9)(100)}}{2(-4.9)} \quad \rightarrow \quad t \approx -2.637 \text{ or } 7.739.$$

The negative result is ignored. The ball lands at $t \approx 7.739$ seconds. The range in which the ball travelled from the base is found by evaluating the *x*-component of displacement by this t:

$$\text{Range} = 43.301(7.739) = 335.106 \text{ m.}$$

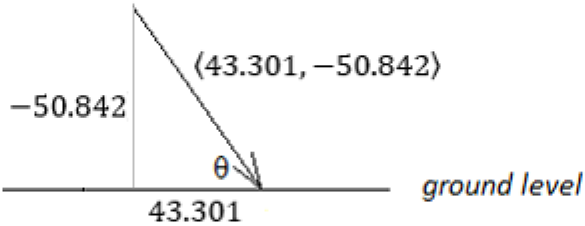
c) We find the velocity at $t = 7.739$ seconds:

$$\mathbf{v}(7.739) = \langle 43.301, -9.8(7.739) + 25\rangle = \langle 43.301, -50.842\rangle.$$

The impact speed of the ball is the magnitude of this vector:

$$\begin{aligned}\text{Speed} = |\langle 43.301, -50.842\rangle| &= \sqrt{43.301^2 + (-50.842)^2} \\ &\approx 66.702 \text{ m/s.}\end{aligned}$$

d) To find the angle at which the ball impacts the ground, sketch a diagram to be sure that the components of velocity are properly in place:



The angle of impact is $\theta = \tan^{-1}\frac{-50.842}{43.301} = -49.6$ degrees. The negatives can be ignored, so that the impact angle is 49.6 degrees.

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Example 22.2: An airplane is flying horizontally 5,000 meters above flat ground at a velocity of 300 kilometers per hour. An object is released below the airplane and allowed to fall to earth.

a) How long is the object airborne?
b) How far downrange does the object impact the ground?

Solution: We need to convert 300 kilometers per hour into meters per second:

$$\frac{300\text{ km}}{1\text{ hr}} \times \frac{1\text{ hr}}{3600\text{ sec}} \times \frac{1000\text{ m}}{1\text{ km}} = \frac{(300)(1000)}{3600} = \frac{250}{3}\ \frac{\text{meters}}{\text{second}}.$$

The initial velocity is therefore $\mathbf{v}(0) = \langle v_x, v_y\rangle = \langle \frac{250}{3}, 0\rangle$, where the 0 in the y-component suggests that the object was merely released with no initial push in the vertical direction. Integrating the acceleration vector $\mathbf{a}(t) = \langle 0, -9.8\rangle$, the velocity vector is

$$\mathbf{v}(t) = \langle v_x, -9.8t + v_y\rangle = \left\langle \frac{250}{3}, -9.8t\right\rangle, \qquad t \geq 0.$$

For initial position, the object was released 5000 meters above the ground suggests that $r_y = 5000$, while $r_x = 0$. Therefore, the displacement function is

$$\mathbf{r}(t) = \left\langle \frac{250}{3}t, -4.9t^2 + 5000\right\rangle, \qquad t \geq 0.$$

a) The object is airborne for the time needed for the y-component of displacement to equal 0 meters, the height of the ground. Thus, we have

$$-4.9t^2 + 5000 = 0.$$

Solving, we get $t = \sqrt{\frac{5000}{4.9}} \approx 31.944$ seconds.

b) The range is found by evaluating the *x*-component of displacement at $t = 31.944$ seconds. We find that the object travelled $\frac{250}{3}(31.944) =$ 2,662 meters.

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Example 22.3: A cannon is angled at 25 degrees above the flat ground. A cannonball is shot from this cannon, and it lands 50 meters downrange. Assume the cannonball left the cannon 1 meter above the ground.

a) What was the initial speed of the cannonball?
b) How many seconds was the cannonball in the air?
c) What was the maximum height of the cannonball?

Solution: Let v_0 represent the initial speed. Using trigonometry, the *x* and *y* components of the initial velocity are $v_x = v_0 \cos 25°$ and $v_y = v_0 \sin 25°$. Therefore, the velocity vector is

$$\mathbf{v}(t) = \langle v_0 \cos 25°, -9.8t + v_0 \sin 25° \rangle, \qquad t \geq 0.$$

By placing the origin directly below where the cannonball began its trajectory, we have $r_x = 0$ and $r_y = 1$. The displacement vector is

$$\mathbf{r}(t) = \langle (v_0 \cos 25°)t, -4.9t^2 + (v_0 \sin 25°)t + 1 \rangle, \qquad t \geq 0.$$

a) The cannonball landed 50 meters downrange, so the *x*-component of displacement is set equal to 50, and we can find an expression for t in terms of v_0:

$$(v_0 \cos 25°)t = 50; \qquad \text{so that} \qquad t = \frac{50}{v_0 \cos 25°}.$$

When the cannonball landed, the *y*-component of displacement is 0 meters, so we have $-4.9t^2 + (v_0 \sin 25°)t + 1 = 0$. Mow substitute $t = \frac{50}{v_0 \cos 25°}$ into this equation, simplify, and solve for v_0:

$$-4.9\left(\frac{50}{v_0 \cos 25°}\right)^2 + (v_0 \sin 25°)\left(\frac{50}{v_0 \cos 25°}\right) = -1$$

$$\frac{-12{,}250}{v_0^2 \cos^2 25°} + \frac{50 \sin 25°}{\cos 25°} = -1$$

Multiply by the common denominator, $v_0^2 \cos^2 25°$ to clear fractions:

$$-12{,}250 + 50v_0^2 \sin 25° \cos 25° = -v_0^2 \cos^2 25°.$$

Collect terms:

$$v_0^2(50 \sin 25° \cos 25° + \cos^2 25°) = 12{,}250.$$

Isolate v_0:

$$v_0 = \left(\frac{12{,}250}{50 \sin 25° \cos 25° + \cos^2 25°}\right)^{1/2} \approx 24.766\ \frac{\text{meters}}{\text{second}}.$$

The cannonball left the cannon at a speed of about 24.766 meters per second.

b) Since $t = \frac{50}{v_0 \cos 25°}$ and $v_0 = 24.766$, we have $t = \frac{50}{(24.766)\cos 25°} \approx$ 2.228 seconds. This is the time the cannonball was in the air.

c) The maximum height occurs when the *y*-component of velocity is 0:

$$-9.8t + 24.766 \sin 25° = 0 \quad \rightarrow \quad t = \frac{24.766 \sin 25°}{9.8} \approx 1.068 \text{ seconds.}$$

This is substituted into the *y*-component of displacement:

$$-4.9(1.068)^2 + (24.766 \sin 25°)(1.068) + 1 = 5.59 \text{ meters.}$$

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Example 22.4: An object is projected from the top of a 50-meter-tall cliff. It remains in the air for 8 seconds, landing 250 meters downrange. Find the following:

a) The maximum height above ground that the object achieved;
b) The initial speed and angle above the horizontal that the object was released.

Solution: Starting with $\mathbf{a}(t) = \langle 0, -9.8\rangle$, we develop vector-valued functions for $\mathbf{v}(t)$ and $\mathbf{r}(t)$:

$$\mathbf{v}(t) = \langle v_x, -9.8t + v_y\rangle \ \& \ \mathbf{r}(t) = \langle v_x t + r_x, -4.9t^2 + v_y t + r_y\rangle, \qquad t \geq 0.$$

where $\langle v_x, v_y\rangle$ are the initial components of the velocity when the object is released, and $\langle r_x, r_y\rangle$ is its initial position. By setting the origin at the base of the cliff directly below the point where the object was released, we have $r_x = 0$ and $r_y = 50$. Thus, we have

$$\mathbf{r}(t) = \langle v_x t, -4.9t^2 + v_y t + 50\rangle.$$

At $t = 8$ seconds, the object lands 250 meters downrange, at which time its vertical component of position will be 0. Thus, we examine the components of $\mathbf{r}(t)$:

$$\begin{aligned} v_x(8) &= 250 \\ -4.9(8)^2 + v_y(8) + 50 &= 0. \end{aligned}$$

From the first equation, we have $v_x = \frac{250}{8} = 31.25$ meters per second, and from the second equation, we have $v_y = \frac{(4.9)(64)-50}{8} = 32.95$ meters per second. We can now identify the velocity and position vectors precisely:

$$\mathbf{v}(t) = \langle 31.25, -9.8t + 32.95\rangle; \quad \text{and} \quad \mathbf{r}(t) = \langle 31.25t, -4.9t^2 + 32.95t + 50\rangle.$$

With the velocity and position vectors now established, we can now address the questions:

a) The object reaches its maximum height when the y-component of velocity is 0, or when $-9.8t + 32.95 = 0$, giving $t = \frac{32.95}{9.8} \approx 3.362$ seconds. This is then evaluated into the y-component of position. The object's maximum height is

$$-4.9(3.362)^2 + 32.95(3.362) + 50 \approx 105.393 \text{ meters.}$$

b) The initial velocity is $\mathbf{v}(0) = \langle 31.25, 32.95\rangle$. Thus, the initial speed of the object is

$$\sqrt{31.25^2 + 32.95^2} \approx 45.4 \text{ meters per second,}$$

and its initial angle above the horizontal is

$$\theta = \arctan\left(\frac{v_y}{v_x}\right) = \arctan\left(\frac{32.95}{31.25}\right) \approx 46.52°.$$

Example 22.5: An object is propelled off a building that is 75 m high, the object propelled at an initial angle of 40° to the horizontal. It reaches a maximum height after 1.75 seconds. How far downrange does the object land, assuming the ground below to be flat?

Solution: We start with velocity $\mathbf{v}(t) = \langle v_x, -9.8t + v_y \rangle$ and position $\mathbf{r}(t) = \langle v_x t + r_x, -4.9t^2 + v_y t + r_y \rangle$. Assuming a starting position of $\langle v_x, v_y \rangle = \langle 0, 75 \rangle$, we can further refine the position vector function as $\mathbf{r}(t) = \langle v_x t, -4.9t^2 + v_y t + 75 \rangle$.

From the velocity vector, the object reaches its maximum height when its *y*-component is 0. Thus, we have $-9.8(1.75) + v_y = 0$, which gives $v_y = 17.15$ meters per second.

From this, we can reconstruct the initial velocity components. We have a right triangle with an angle of 40° and an opposite leg of 17.15 meters per second. Thus, the adjacent leg is given by $\tan 40° = \frac{\text{opposite}}{\text{adjacent}} = \frac{17.15}{v_x}$, so that $v_x = \frac{17.15}{\tan 40°} = 20.439$ meters per second.

This allows us to completely define the position vector, $\mathbf{r}(t) = \langle 20.439t, -4.9t^2 + 17.15t + 75 \rangle, t \geq 0$. The object impacts the ground when the *y*-component of $\mathbf{r}(t)$ is 0:

$$-4.9t^2 + 17.15t + 75 = 0.$$

Using the quadratic formula, we get two results:

$$t = \frac{-(17.15) \pm \sqrt{(17.15)^2 - 4(-4.9)(75)}}{2(-4.9)} \quad \rightarrow \quad t \approx -2.536, t \approx 6.036.$$

The negative result is ignored. The positive result, 6.036 seconds, is the time after release that the object impacts the ground. Its distance from the base of the building is given by the *x*-component of position:

$$20.439(6.036) = 123.37.$$

Thus, the object travelled a total of about 123.37 meters in the horizontal direction before impacting the ground.

Example 22.6: A golf ball is hit from the ground by an astronaut on a distant planet. It reaches a maximum height of 50 meters after 4 seconds of flight. What is the gravitational constant on this planet?

Solution: We start with $\mathbf{a}(t) = \langle 0, -a \rangle$, then develop $\mathbf{v}(t)$ and $\mathbf{r}(t)$:

$$\mathbf{v}(t) = \langle v_x, -at + v_y \rangle \quad \text{and} \quad \mathbf{r}(t) = \langle v_x t + r_x, -\frac{a}{2}t^2 + v_y t + r_y \rangle.$$

Assume that the initial position is $\langle v_x, v_y \rangle = \langle 0,0 \rangle$. Thus, position is given by $\mathbf{r}(t) = \langle v_x t, -\frac{a}{2}t^2 + v_y t \rangle, t \geq 0$.

When the ball reaches the maximum height, its vertical component of velocity is 0, while the vertical component of position is 50. This creates a pair of equations:

$$\begin{aligned} -at + v_y &= 0 \\ -\frac{a}{2}t^2 + v_y t &= 50. \end{aligned}$$

When $t = 4$, we obtain $-4a + v_y = 0$ and $-\frac{a}{2}(4)^2 + 4v_y = 50$. From the first equation, we have

$$v_y = 4a,$$

and from the second equation, we have

$$-8a + 4(4a) = 50.$$

Solving for a, we obtain

$$\begin{aligned} -8a + 16a &= 50 \\ 8a &= 50 \\ a &= \frac{50}{8} = 6.25. \end{aligned}$$

Evidently, objects on this planet fall at a rate of 6.25 meters per second^2.