24. Partial Differentiation

The derivative of a single variable function, $\frac{d}{dx}f(x)$, always assumes that the independent variable is increasing in the usual manner. Visually, the derivative's value at a point $x = a$ is the slope of the tangent line of $y = f(x)$ at $x = a$, and the slope's value only "makes sense" if x increases to the right, as viewed on a standard xy-axis system. This is the *direction* of $\frac{d}{dx}f(x)$. On the real line representing the independent variable x, there are just two directions in which x can vary: to the right or to the left. However, with a multivariable function $z = f(x, y)$, the number of possible directions in which the independent variables can vary (together) can be infinite.

Thus, when finding the instantaneous rate of change between the dependent variable and one of the independent variables of a multivariable function $z = f(x, y)$, we must specify clearly a direction in which we are comparing this rate of change.

For example, if given a function $z = f(x, y)$, and assuming for now that its graph is continuous everywhere and smooth, in that it lacks corners and folds, then there are two possible "convenient" directions in which to calculate an instantaneous rate of change: the positive x direction, or the positive y direction. (There are actually infinitely-many directions. This is discussed in Section 26.)

The instantaneous rate of change of z with respect to x is called **the partial derivative of z with respect to x**, and is written

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial x}, \quad \text{or informally as } z_x \text{ or } f_x.$$

In this case, the variable y is held constant. It does not vary.

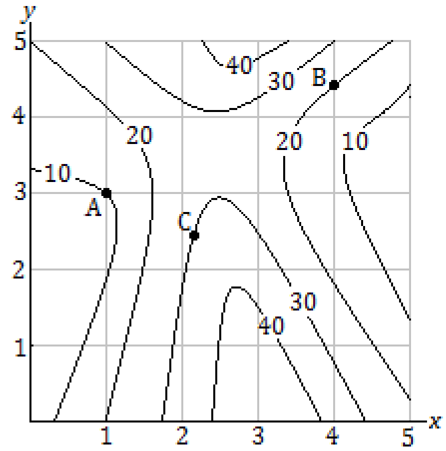
Similarly, the instantaneous rate of change of z with respect to y is called **the partial derivative of z with respect to y**, and is written

$$\frac{\partial z}{\partial y} \quad \text{or} \quad \frac{\partial f}{\partial y}, \quad \text{or informally as } z_y \text{ or } f_y.$$

Here, x is now held constant.

♦ • ♦ • ♦ • ♦ • ♦

Example 24.1: Use the contour map below, representing $z = f(x, y)$, to answer the questions that follow. Assume that f is smooth and continuous.



a) Use a convenient nearby point to estimate the slope of a tangent line drawn at *A*, in the positive *x* direction, then estimate the slope of a tangent line at this same point, now in the positive *y* direction.
b) Use a convenient nearby point to estimate the slope of a tangent line drawn at *B*, in the positive *x* direction, then estimate the slope of a tangent line at this same point, now in the positive *y* direction.
c) Estimate the sign of the slope of a tangent line drawn at *C*, in the positive *x* direction, then estimate the sign of the slope of a tangent line at this same point, now in the positive *y* direction.

Solution:

a) Observe that *A* is given by the ordered triple (1, 3, 10). A convenient nearby point in the positive *x* direction is (1.6, 3, 20). When moving in the positive *x* direction, variable *y* remains constant. Thus, a reasonable estimation of the slope of a tangent line drawn at *A* in the positive *x* direction is $\frac{\partial}{\partial x} f(A) \approx \frac{20-10}{1.6-1} = \frac{10}{0.6} = 16.7$. Note that $\frac{\partial}{\partial x} f(A)$ is positive. When at *A*, and moving in the positive *x* direction, we see that we would be walking toward higher ground, as shown by the *z* = 20 contour.

   In a similar way, we use the ordered triple (1, 4.2, 20) as a convenient point in the positive y direction. Thus, a reasonable estimation of the slope of a tangent line drawn at *A* in the positive *y* direction is $\frac{\partial}{\partial y} f(A) \approx \frac{20-10}{4.2-3} = \frac{10}{1.2} = 8.3$. Note that when moving in the positive *y* direction, variable *x* remains constant.

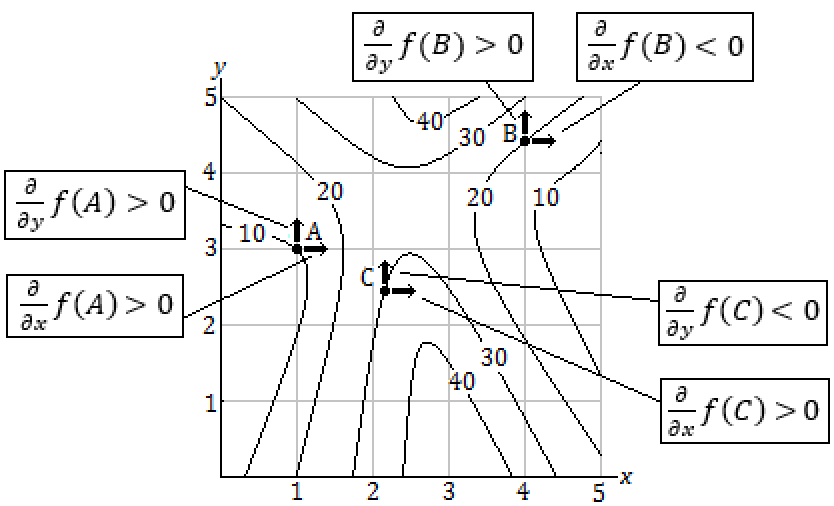
b) Point B is (4, 4.5, 20). A convenient point in the positive x direction is (5, 4.5, 10). Thus, a reasonable estimation of the slope of a tangent line drawn at B in the positive x direction is $\frac{\partial}{\partial x} f(B) \approx \frac{10-20}{5-4} = -10$. Here, moving in the positive x direction means a negative (downward) change in z.

In the y direction, we use (4, 5, 30), and a reasonable estimation of the slope of a tangent line drawn at B in the positive y direction is $\frac{\partial}{\partial y} f(B) \approx \frac{30-20}{5-4.5} = \frac{10}{0.5} = 20$. Moving in the positive y directions means a positive (upward) change in z.

c) Rather than choosing points nearby to C, we will study the contour map and make a judgement of the signs of $\frac{\partial}{\partial x} f(C)$ and $\frac{\partial}{\partial y} f(C)$.

The z-value at C is 30. Moving a small distance in the positive x direction would place us in a region where $30 < z < 40$. Thus, this means that any immediate movement off of C in the positive x direction would result in an increase in the z value. Thus, we can conclude that $\frac{\partial}{\partial x} f(C)$ is positive.

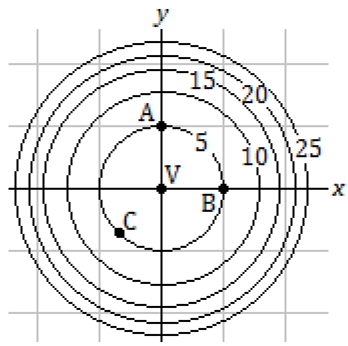
Moving a small distance in the positive y direction puts us in a region where $20 < z < 30$. This means that any immediate movement off of C in the positive y direction would result in a decrease in the z value. Thus, we can conclude that $\frac{\partial}{\partial y} f(C)$ is negative.



Small arrows are drawn in the positive x and y directions to help suggest the signs of the partial derivatives in these directions, at points A, B and C.

◆ • ◆ • ◆ • ◆ • ◆

Example 24.2: Use the contour map below, representing a paraboloid $z = f(x,y)$ that opens in the positive z direction, to answer the questions that follow. Assume that f is smooth and continuous, and that the vertex V is at the origin and is the minimum point. Determine signs of the partial derivatives of f with respect to x and with respect to y at points A, B, C and V.



Solution: As in the previous example, we can sketch in small arrows at each point to help suggest the sign of the partial derivative in a particular direction. *But this method must be used carefully*!

We assume that if a surface representing f is smooth and continuous, then the partial derivatives of f with respect to x and with respect to y are 0 at all minimum, maximum and saddle points.

Thus, we can immediately identify the signs of the partial derivatives of f with respect to x and with respect to y at the vertex V. Since V is a minimum, then $\frac{\partial}{\partial x} f(V) = 0$ and $\frac{\partial}{\partial y} f(V) = 0$.

For A, we note that any movement in the positive y direction will mean a positive change in z. Thus, $\frac{\partial}{\partial y} f(A) > 0$. However, note that movement in the positive x direction is tangential to the level curve. In such cases, the change in z with respect to x is 0. Thus, $\frac{\partial}{\partial x} f(A) = 0$.

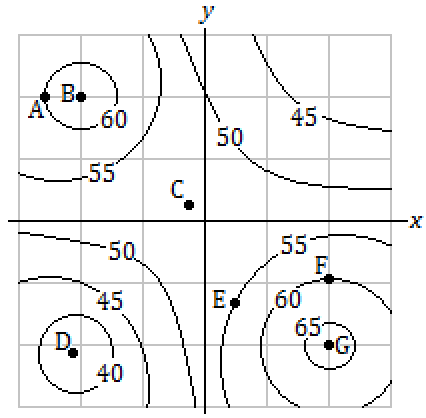
For B, we have $\frac{\partial}{\partial x} f(B) > 0$ and $\frac{\partial}{\partial y} f(B) = 0$, since movement in the y direction is tangential to the level curve.

For C, movements in the x or the y direction are not tangential to the level curve. In both cases, z will decrease in value, so that $\frac{\partial}{\partial x} f(C) < 0$ and $\frac{\partial}{\partial y} f(C) < 0$.

♦ • ♦ • ♦ • ♦ • ♦

On a contour map representing the surface of a smooth and continuous function f, the values of the partial derivatives of f with respect to x and with respect to y are 0 at all minimum, maximum and saddle points. If movement in the x or y direction happens to be tangential to the contour at a point, then the value of the partial derivative of f with respect to the x or y direction is 0. That is, tangential movement along a level curve always means no change in z.

Example 24.3: The contour map below represents the surface of a smooth and continuous function $z = g(x, y)$. Assume that points *B*, *C*, *D* and *G* are minimum, maximum or saddle points. State the sign (positive, negative or zero) of the partial derivative of g with respect to *x* and with respect to *y*, at each of the points *A* through *G*.



Solution:

- For point *A*, we have $\frac{\partial}{\partial x} g(A) > 0$ and $\frac{\partial}{\partial y} g(A) = 0$.
- For point *B*, we have $\frac{\partial}{\partial x} g(B) = 0$ and $\frac{\partial}{\partial y} g(B) = 0$. *B* is a local maximum.
- For point *C*, we have $\frac{\partial}{\partial x} g(C) = 0$ and $\frac{\partial}{\partial y} g(C) = 0$. *C* is a saddle point.
- For point *D*, we have $\frac{\partial}{\partial x} g(D) = 0$ and $\frac{\partial}{\partial y} g(D) = 0$. *D* is a local minimum.
- For point *E*, we have $\frac{\partial}{\partial x} g(E) > 0$ and $\frac{\partial}{\partial y} g(E) < 0$.
- For point *F*, we have $\frac{\partial}{\partial x} g(F) = 0$ and $\frac{\partial}{\partial y} g(F) < 0$.
- For point *G*, we have $\frac{\partial}{\partial x} g(G) = 0$ and $\frac{\partial}{\partial y} g(G) = 0$. *G* is a local maximum.

Note that *A* is tangent to the contour in the *y* direction, and that *F* is tangent to the contour in the *x* direction.

♦ • ♦ • ♦ • ♦ • ♦

The rules for partial differentiation are identical to single variable integration. The Product Rule, Quotient Rule and Chain Rule are all used as necessary.

Example 24.4: Given $z = f(x, y) = x^2y + 3x^3y^4 + 2x - 4y$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: When finding $\frac{\partial z}{\partial x}$, treat the *y* as a constant. If it is in a term by itself, then the whole term is treated as a constant. If it is connected to *x* through multiplication, then it is treated as a coefficient. Thus, we have

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}(x^2y + 3x^3y^4 + 2x - 4y) \\
&= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial x}(3x^3y^4) + \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial x}(4y) \\
&= (2x)y + 3(3x^2)y^4 + 2(1) - 0 \\
&= 2xy + 9x^2y^4 + 2.
\end{aligned}$$

Similarly, to find $\frac{\partial z}{\partial y}$, we treat *x* as a constant or a coefficient:

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}(x^2y + 3x^3y^4 + 2x - 4y) \\
&= \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial y}(3x^3y^4) + \frac{\partial}{\partial y}(2x) - \frac{\partial}{\partial y}(4y) \\
&= x^2(1) + 3x^3(4y^3) + 0 - 4(1) \\
&= x^2 + 12x^3y^3 - 4.
\end{aligned}$$

♦ • ♦ • ♦ • ♦ • ♦

Example 24.5: Given $z = g(x, y) = xye^y$. Find $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$.

Solution: For $\frac{\partial g}{\partial x}$, the factors ye^y are attached to *x* by multiplication and are treated as a coefficient of *x*. Thus,

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(xye^y) = (1)ye^y = ye^y, \qquad \text{where } \frac{\partial}{\partial x}x = 1.$$

For $\frac{\partial g}{\partial y}$, the *x* is now treated as a coefficient, and the Product Rule of differentiation is used:

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(xye^y) = xy\frac{\partial}{\partial y}(e^y) + e^y\frac{\partial}{\partial y}(xy) = xye^y + xe^y.$$

♦ • ♦ • ♦ • ♦ • ♦

Example 24.6: Given $z = x^3 \sin(x^2y^3)$. Find z_x and z_y.

Solution: For z_x, note that *x* is present in two factors attached by multiplication. Thus, we use the Product Rule of differentiation and the Chain Rule:

$$\begin{aligned} z_x &= x^3(\cos(x^2y^3)\, 2xy^3) + 3x^2 \sin(x^2y^3) \\ &= 2x^4y^3 \cos(x^2y^3) + 3x^2 \sin(x^2y^3). \end{aligned}$$

For z_y, we do not need the Product Rule, treating the x^3 as a coefficient of the sine operator. However, we do need the Chain Rule:

$$\begin{aligned} z_y &= x^3 \cos(x^2y^3)\, x^2(3y^2) \\ &= 3x^5y^2 \cos(x^2y^3). \end{aligned}$$

♦ • ♦ • ♦ • ♦ • ♦

Partial differentiation can be used for functions with more than two variables.

Example 24.7: The function $A(p, r, t) = p(1 + r)^t$ gives the future value *A* of *p* dollars invested at an annual percentage rate *r*, compounded annually, after *t* years. Find A_p, A_t and A_r.

Solution: To find A_p, note that $(1 + r)^t$ is treated as a constant multiplier to *p*. Since $\frac{\partial}{\partial p}(p) = 1$, we have

$$A_p = \frac{\partial}{\partial p}(p(1 + r)^t) = (1)(1 + r)^t = (1 + r)^t.$$

To find A_t, we use the differentiation rule for exponentials, $\frac{d}{dx}(a^x) = a^x \ln a$. Thus, we have

$$A_t = \frac{\partial}{\partial t}(p(1 + r)^t) = p(1 + r)^t \ln(1 + r).$$

To find A_r, note that *p* and *t* are constants. Thus, we can use the Power Rule of differentiation:

$$A_r = \frac{\partial}{\partial r}(p(1 + r)^t) = pt(1 + r)^{t-1}.$$

♦ • ♦ • ♦ • ♦ • ♦

Higher-Order Partial Derivatives & Clairaut's Theorem

Partial differentiation can also be used to find second-order derivatives, and so on. Suppose $z = f(x, y)$ is given. There are two first partial derivatives,

$$f_x = \frac{\partial f}{\partial x} \quad \text{and} \quad f_y = \frac{\partial f}{\partial y}.$$

Each partial derivative is itself a function of two variables. Thus, each has two partial derivatives of its own. For example, $f_x(x, y)$ has two partial derivatives:

$$(f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}.$$

Similarly, $f_y(x, y)$ has two partial derivatives:

$$\left(f_y\right)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} \quad \text{and} \quad \left(f_y\right)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y}.$$

Usually, second derivatives are noted by using subscripts without parentheses. Thus,

$$f_{xx} = (f_x)_x, \;\; f_{yy} = \left(f_y\right)_y, \;\; f_{xy} = (f_x)_y \;\; \text{and} \;\; f_{yx} = \left(f_y\right)_x.$$

Second derivatives such as f_{xx} and f_{yy} are informally called *homogeneous* second derivatives, while f_{xy} and f_{yx} are called *mixed* second derivatives.

Under "typical" circumstances, *e.g.* the function f being smooth and continuous, and twice-differentiable over its relevant domain, the mixed second derivatives will be equal:

$$f_{xy} = f_{yx} \quad (\text{Clairaut's Theorem}).$$

As one might expect, second derivatives of a smooth and continuous function offer insight to the concavity of the function.

Higher-order derivatives are found in a similar manner. For example,

$$f_{xxy} = ((f_x)_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} f(x, y)\right)\right) = \frac{\partial^3 f}{\partial y\, \partial x^2}.$$

◆ • ◆ • ◆ • ◆ • ◆

Example 24.8: Given $z = f(x, y) = x^2y + 3x^3y^4 + 2x - 4y$. Find f_{xx}, f_{yy}, f_{xy} and f_{yx}.

Solution: From a previous example, we found the two first partial derivatives:

$$f_x(x, y) = 2xy + 9x^2y^4 + 2 \quad \text{and} \quad f_y(x, y) = x^2 + 12x^3y^3 - 4.$$

Thus, we have

$$f_{xx} = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial}{\partial x}(2xy + 9x^2y^4 + 2) = 2y + 18xy^4$$

and

$$f_{yy} = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial}{\partial y}(x^2 + 12x^3y^3 - 4) = 36x^3y^2.$$

Furthermore, we have

$$f_{xy} = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial}{\partial y}(2xy + 9x^2y^4 + 2) = 2x + 36x^2y^3$$

and

$$f_{yx} = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial}{\partial x}(x^2 + 12x^3y^3 - 4) = 2x + 36x^2y^3.$$

Note that $f_{xy} = f_{yx}$.

♦ • ♦ • ♦ • ♦ • ♦

Example 24.9: Given $a = b^3c^4d^5$. Show that $a_{bcd} = a_{dbc}$.

Solution: We find successive partial derivatives by reading the subscripts left to right. For example, $a_{bcd} = ((a_b)_c)_d$. We have

$$a_b = 3b^2c^4d^5, \quad \text{so that} \quad a_{bc} = 12b^2c^3d^5, \quad \text{and finally} \quad a_{bcd} = 60b^2c^3d^4.$$

Similarly,

$$a_d = 5b^3c^4d^4, \quad \text{so that} \quad a_{db} = 15b^2c^4d^4, \quad \text{and finally} \quad a_{dbc} = 60b^2c^3d^4.$$

There are six orderings in which to take the derivative of *a* with respect to *b*, *c* and *d* in any order. We have found two. You should find the other four and verify all are equal.

♦ • ♦ • ♦ • ♦ • ♦

Example 24.10: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, where $f(x,y) = \int_x^y 3t^2\,dt$.

Solution: Defining functions as integrals is not uncommon. In this case, we can antidifferentiate the integrand, and evaluate at the limits of integration:

$$f(x,y) = \int_x^y 3t^2\,dt = [t^3]_x^y = y^3 - x^3.$$

Taking partial derivatives, we have,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y^3 - x^3) = -3x^2 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y^3 - x^3) = 3y^2.$$

Note that the results look similar to the original integrand. Was it necessary to do the antidifferentiation step? See the next example.

♦ • ♦ • ♦ • ♦ • ♦

Example 24.11: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, where $f(x,y) = \int_x^y \sqrt{t^4 - 2t + 7}\,dt$.

Solution: Repeating the steps of the previous example leads to a seemingly insurmountable problem: the integrand does not antidifferentiate "conveniently" into common elementary functions. For now, define $H(t)$ to be the antiderivative of $\sqrt{t^4 - 2t + 7}$. We cannot determine $H(t)$, but we know its derivative is $\sqrt{t^4 - 2t + 7}$. Thus, we have

$$f(x,y) = \int_x^y \sqrt{t^4 - 2t + 7}\,dt = [H(t)]_x^y = H(y) - H(x).$$

Thus,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\big(H(y) - H(x)\big) = -\sqrt{x^4 - 2x + 7},$$

where $\frac{\partial}{\partial x}\big(H(y)\big) = 0$ and where $\frac{\partial}{\partial x}\big(H(x)\big)$ is the derivative of $H(t)$ with *x* in place of *t*.

In a similar manner,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\big(H(y) - H(x)\big) = \sqrt{y^4 - 2y + 7}.$$

♦ • ♦ • ♦ • ♦ • ♦