12. Lines in R^3

Given a point $P_0 = (x_0, y_0, z_0)$ and a direction vector $\mathbf{v}_1 = \langle a, b, c\rangle$ in R^3, a line L that passes through P_0 and is parallel to $\mathbf{v}$ is written parametrically as a function of t:

$$x(t) = x_0 + at, \qquad y(t) = y_0 + bt, \qquad z(t) = z_0 + ct$$

Using vector notation, the same line is written

$$\begin{aligned}\langle x, y, z\rangle &= \mathbf{v}_0 + t\mathbf{v}_1 \\ &= \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct\rangle,\end{aligned}$$

where $\mathbf{v}_0$ is the vector whose head is located at $P_0 = (x_0, y_0, z_0)$.

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Example 12.1: Find the parametric equation of a line passing through $P_0 = (2, -1, 3)$ and parallel to the vector $\mathbf{v}_1 = \langle 5,8,-4\rangle$.

Solution: The line is represented parametrically by

$$x(t) = 2 + 5t, \qquad y(t) = -1 + 8t, \qquad z(t) = 3 - 4t,$$

or in vector notation as

$$\langle x, y, z\rangle = \langle 2, -1,3\rangle + t\langle 5,8,-4\rangle = \langle 2 + 5t, -1 + 8t, 3 - 4t\rangle.$$

Note that when $t = 0$, we obtain the vector $\langle 2, -1,3\rangle$. If the foot of this vector is placed at the origin, then its head is the ordered triple $P_0 = (2, -1, 3)$

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A **line segment** from a point $P_0 = (x_0, y_0, z_0)$ to a point $P_1 = (x_1, y_1, z_1)$ over $a \le t \le b$ has the form

$$\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + \frac{t - a}{b - a}\langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle,$$

Note that $\langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$ is the direction vector of the line.

Example 12.2: Find the parametric equation of the line segment from $P_0 = (4, 2, -1)$ to $P_1 = (7, -3, -2)$.

Solution: The direction vector $\mathbf{v}_1$ is found by subtracting P_0 from P_1:

$$\begin{aligned}\mathbf{v}_1 &= P_1 - P_0 \\ &= \langle 7-4, -3-2, -2-(-1)\rangle \\ &= \langle 3, -5, -1\rangle.\end{aligned}$$

Thus, the line can be written

$$\begin{aligned}\langle x, y, z\rangle &= \langle 4,2,-1\rangle + t\langle 3,-5,-1\rangle \\ &= \langle 4+3t, 2-5t, -1-t\rangle, \qquad \text{for } 0 \le t \le 1.\end{aligned}$$

The bounds are such that $t = 0$ gives $P_0 = (4, 2, -1)$ and $t = 1$ gives $P_1 = (7, -3, -2)$. Since there is a direction implied by increasing t, this is called a *directed line segment*.

When a line segment between two points is constructed in this manner, the bounds on t are always $0 \le t \le 1$.

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Example 12.3: Find the parametric equation of the line segment connecting $P_0 = (4, 2, -1)$ and $P_1 = (7, -3, -2)$ such that $t = 0$ gives P_0 and that $t = 5$ gives P_1.

Solution: The difference in t-values is $b - a = 5 - 0 = 5$. Thus, the line segment is

$$\begin{aligned}\langle x, y, z\rangle &= \langle 4,2,-1\rangle + \frac{t}{5}\langle 3,-5,-1\rangle \\ &= \langle 4,2,-1\rangle + t\left\langle \frac{3}{5}, -1, -\frac{1}{5}\right\rangle \\ &= \left\langle 4 + \frac{3}{5}t, 2 - t, -1 - \frac{1}{5}t\right\rangle, \qquad \text{for } 0 \le t \le 5.\end{aligned}$$

Note that $t = 0$ gives P_0 and that $t = 5$ gives P_1.

Example 12.4: Find the parametric equation of a line segment connecting $P_0 = (4, 2, -1)$ and $P_1 = (7, -3, -2)$ such that $t = 4$ gives P_0 and $t = 11$ gives P_1.

Solution: The starting point occurs when $t = 4$, indicating in a horizontal shift. The difference in t-values is $b - a = 11 - 4 = 7$. Thus, the line segment is

$$\begin{aligned}
\langle x, y, z\rangle &= \langle 4,2,-1\rangle + \frac{t-4}{11-4}\langle 3,-5,-1\rangle \\
&= \langle 4,2,-1\rangle + \left\langle \frac{3(t-4)}{7}, -\frac{5(t-4)}{7}, -\frac{1(t-4)}{7}\right\rangle \\
&= \langle 4,2,-1\rangle + \left\langle \frac{3}{7}t - \frac{12}{7}, -\frac{5}{7}t + \frac{20}{7}, -\frac{1}{7}t + \frac{4}{7}\right\rangle \quad \left\{\text{Clearing fractions}\right. \\
&= \left\langle 4 - \frac{12}{7}, 2 + \frac{20}{7}, -1 + \frac{4}{7}\right\rangle + \left\langle \frac{3}{7}t, -\frac{5}{7}t, -\frac{1}{7}t\right\rangle \quad \left\{\text{Combining constants}\right. \\
&= \left\langle \frac{16}{7} + \frac{3}{7}t, \frac{34}{7} - \frac{5}{7}t, -\frac{3}{7} - \frac{1}{7}t\right\rangle, \qquad \text{for } 4 \le t \le 11.
\end{aligned}$$

Note that $t = 4$ gives P_0 and that $t = 11$ gives P_1.

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> When more than one line is being considered,
> use different parameter variables.

Example 12.5: Let L_1: $\langle x, y, z\rangle = \langle 1,2,5\rangle + t\langle 2,4,-3\rangle$ and L_2: $\langle x, y, z\rangle = \langle 6,1,-2\rangle + s\langle 4,8,-6\rangle$ be two lines defined parametrically.

a) Are lines L_1 and L_2 parallel?
b) Are lines L_1 and L_2 the same line?

Solution:

a) The direction vector for line L_1 is $\mathbf{v}_1 = \langle 2,4,-3\rangle$ and the direction vector for line L_2 is $\mathbf{v}_2 = \langle 4,8,-6\rangle$. Since $\mathbf{v}_2 = 2\mathbf{v}_1$ (or equivalently, $\mathbf{v}_1 = \frac{1}{2}\mathbf{v}_2$), the two vectors are parallel. Thus, so are the lines.

b) Choose a point from one line and show that the other line passes through it. In this example, choose point $P_0 = (1,2,5)$ from line L_1. Does L_2 pass through $P_0 = (1,2,5)$? We substitute the coordinates in P_0 for x, y and z, and attempt to solve for a unique value of s that would indicate line L_2 passes through a point in line L_1:

$$1 = 6 + 4s, \qquad 2 = 1 + 8s, \qquad 5 = -2 - 6s.$$

From the first equation, we get $s = -5/4$, but from the second equation, we get $s = 1/8$. Since s is not unique, we conclude it is impossible that L_2 passes through $P_0 = (1,2,5)$. Thus, lines L_1 and L_2 represent two different parallel lines.

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Example 12.6: Show that the lines L_1: $\langle 2,3,0\rangle + t\langle 1,-2,5\rangle$ and L_2: $\langle 6,-5,20\rangle + s\langle -3,6,-15\rangle$ are the same line.

Solution: Their direction vectors are $\mathbf{v}_1 = \langle 1,-2,5\rangle$ and $\mathbf{v}_2 = \langle -3,6,-15\rangle$. Since $\mathbf{v}_2 = -3\mathbf{v}_1$, the two lines are parallel. Had the direction vectors not been parallel, the parametric equations cannot possibly represent the same line.

Now, choose a point from one line, and see if it is possible to find a unique value for the parameter variable in the other line. From line L_1, choose the point $P_0 = (2,3,0)$ and then attempt to find a unique value for s in L_2:

$$2 = 6 - 3s, \qquad 3 = -5 + 6s, \qquad 0 = 20 - 15s.$$

From the first equation, we get $s = 4/3$. From the second equation, we also get $s = 4/3$, and from the third equation, we get $s = 4/3$. Since we were able to find a unique value s, we conclude that line L_2 passes through a point that is in line L_1. Since the lines are already parallel, this would force the two lines to be **coincident**, that is, the same line.

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Example 12.7: Find the point of intersection of lines L_1: $\langle x,y,z\rangle = \langle 1,2,-1\rangle + t\langle 2,-3,4\rangle$ and L_2: $\langle x,y,z\rangle = \langle 1,8,9\rangle + s\langle 4,-12,-2\rangle$.

Solution: The direction vectors are $\mathbf{v}_1 = \langle 2,-3,4\rangle$ and $\mathbf{v}_2 = \langle 4,-12,-2\rangle$. Since they are not scalar multiples of one another, the two lines are not parallel. To see if they intersect, we set the equations for x equal to one another, and for y, and for z:

$$\begin{aligned} x&: \quad 1 + 2t = 1 + 4s \\ y&: \quad 2 - 3t = 8 - 12s \\ z&: \; -1 + 4t = 9 - 2s. \end{aligned}$$

Simplifying, we have a system of two variables in three equations:

$$\begin{aligned} 2t - 4s &= 0 \\ -3t + 12s &= 6 \\ 4t + 2s &= 10. \end{aligned}$$

One of two things happens: either we find a solution in s and t, in which case there is an intersection point, or we do not find a solution in s and t, in which case there is no intersection point. From the first two equations, we solve a system:

$$\begin{array}{r} 2t-4s=0 \\ -3t+12s=6 \end{array} \xrightarrow{\left(\substack{\text{Multiply}\\ \text{top row by 3}}\right)} \begin{array}{r} 3(2t-4s)=0 \\ -3t+12s=6 \end{array}$$

$$\xrightarrow{\left(\substack{\text{Distribute}\\ \text{then add}}\right)} \begin{array}{r} 6t-12s=0 \\ -3t+12s=6 \end{array} \rightarrow \ 3t=6.$$

Thus, we have $t=2$, and back-substituting, we have $s=1$. Does this solve the third equation? We substitute and simplify:

$$4(2)+2(1)=8+2=10.$$

We get a true statement. We were able to show that when $t=2$, we generate the point $(5,-4,7)$ on line L_1, and when $s=1$, we generate the same point $(5,-4,7)$ on line L_2. Thus, the two lines intersect at this point.

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Two non-parallel and non-intersecting lines in R^3 are called **skew** lines.

Example 12.8: Show that the lines L_1: $\langle x,y,z\rangle=\langle 1,2,-1\rangle+t\langle 2,-3,4\rangle$ and L_2: $\langle x,y,z\rangle=\langle 1,8,5\rangle+s\langle 4,-12,-2\rangle$ are skew lines.

Solution: Note that these are the same lines as from the previous example, except that a small change has been made to the equation for *z* in line L_2. From the previous example, we established that since the direction vectors are not scalar multiples of one another, then the lines are not parallel. Next, we set the equations for *x* equal to one another, and for *y*, and for *z*:

$$\begin{aligned} x\!: &\quad 1+2t=1+4s \\ y\!: &\quad 2-3t=8-12s \\ z\!: &\ -1+4t=5-2s. \end{aligned}$$

This simplifies to

$$\begin{array}{r} 2t-4s=0 \\ -3t+12s=6 \\ 4t+2s=6. \end{array}$$

From the previous example, solving the system formed by the first two equations gave us $t=2$ and $s=1$. However, when substituting these values in the third equation, we get $4(2)+2(1)=6$, which is false. There is no solution of this system in s and t. Thus, the two lines do not intersect, and are skew.