44. Scalar Line Integrals

Let $z = f(x, y)$ be a continuous function (surface) in R^3 and *C* a path on the *xy*-plane. If *C* is parametrized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ for $a \le t \le b$, then the **scalar line integral** of f along *C* is given by

$$\int_C f(x, y)\, ds,$$

where $ds = |\mathbf{r}'(t)|\, dt$. Thus, the integral is in variable t and can be written

$$\int_a^b f\big(x(t), y(t)\big)\ |\mathbf{r}'(t)|\, dt.$$

The value of a scalar line integral is the area of a "sheet" above the path *C* to the surface f.

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Example 44.1: Find $\int_C x^2 y\, ds$, where *C* is the straight line from (2,1) to (6,4).

Solution: Parametrize the path *C* first, noting that $\langle 4,3\rangle$ is the direction vector of the line segment:

$$\mathbf{r}(t) = \langle 2,1\rangle + t\langle 4,3\rangle = \langle 2 + 4t, 1 + 3t\rangle, \qquad \text{for} \quad 0 \le t \le 1.$$

Thus, $\mathbf{r}'(t) = \langle 4,3\rangle$ and $|\mathbf{r}'(t)| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$, so that $ds = |\mathbf{r}'(t)|\, dt = 5\, dt$.

From $\mathbf{r}(t)$, we obtain $x(t) = 2 + 4t$ and $y(t) = 1 + 3t$. These are substituted into the integrand, and simplified:

$$\begin{aligned}
\int_C x^2 y\, ds &= \int_C (2 + 4t)^2 (1 + 3t)\, ds \\
&= \int_C 4(12t^3 + 16t^2 + 7t + 1)\, 5\, dt \\
&= 20 \int_0^1 (12t^3 + 16t^2 + 7t + 1)\, dt \\
&= 20 \left[3t^4 + \frac{16}{3}t^3 + \frac{7}{2}t^2 + t\right]_0^1 \\
&= \frac{770}{3}.
\end{aligned}$$

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Example 44.2: Find $\int_C x\,ds$, where *C* is the arc of the parabola $y = x^2$ from $(-1,1)$ to $(3,9)$.

Solution: Path *C* is parametrized:

$$\mathbf{r}(t) = \langle t, t^2 \rangle, \quad \text{for} \quad -1 \le t \le 3.$$

We have $\mathbf{r}'(t) = \langle 1, 2t \rangle$ and $|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}$, so that $ds = \sqrt{1 + 4t^2}\,dt$. The integrand is now written in terms of t and evaluated:

$$\begin{aligned}
\int_C x\,ds &= \int_{-1}^{3} t\sqrt{1 + 4t^2}\,dt \\
&= \left[\frac{1}{12}(1 + 4t^2)^{3/2}\right]_{-1}^{3} \\
&= \left(\frac{1}{12}(1 + 4(3)^2)^{3/2}\right) - \left(\frac{1}{12}(1 + 4(-1)^2)^{3/2}\right) \\
&= \frac{1}{12}(37^{3/2} - 5^{3/2}) \approx 17.82.
\end{aligned}$$

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In some cases, a numerical method needs to be used to evaluate the integral.

Example 44.3: Find $\int_C x^3 y^2\,ds$, where *C* is the curve $y = x^3$ from (1,1) to (2,8).

Solution: Path *C* is parametrized as:

$$\mathbf{r}(t) = \langle t, t^3 \rangle, \ \text{for} \ 1 \le t \le 2.$$

We have $\mathbf{r}'(t) = \langle 1, 3t^2 \rangle$ and $|\mathbf{r}'(t)| = \sqrt{1^2 + (3t^2)^2} = \sqrt{1 + 9t^4}$. The integrand is now written in terms of t:

$$x^3 y^2\,ds = (t)^3 (t^3)^2 \sqrt{1 + 9t^4}\,dt = t^9 \sqrt{1 + 9t^4}\,dt.$$

The integral is

$$\int_C x^3 y^2\,ds = \int_1^2 t^9 \sqrt{1 + 9t^4}\,dt.$$

Using numerical methods, this integral evaluates to approximately 1029.1 units.

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Example 44.4: The roof of a building is a paraboloid modeled by $= 10 - \frac{1}{8}x^2 - \frac{1}{12}y^2$, with the apex (highest point) above the origin. Assume *x* and *y* represent distances from the origin along the floor (the *xy*-plane) orthogonal to one another, and that all measurements are in meters. A wall is to be built extending from the origin along the line $y = 3x$. The wall will rise from the floor up to the roof. What is the area of this wall?

Solution: The path *C* is the line $y = 3x$, which is parametrized as $\mathbf{r}(t) = \langle t, 3t \rangle$. The lower bound for t is 0. The upper bound of t is where the roof meets the floor along this line. Since $z = 0$ at this point, and since $y = 3x$, we make substitutions and solve for x:

$$\begin{aligned}
0 &= 10 - \frac{1}{8}x^2 - \frac{1}{12}(3x)^2 \\
0 &= 10 - \frac{1}{8}x^2 - \frac{3}{4}x^2 \\
0 &= 10 - \frac{7}{8}x^2.
\end{aligned}$$

Solving for *x*, we get $x = \sqrt{80/7} \approx 3.381$. Since $x = t$ in the parametrization, we have the upper bound for t as $\sqrt{80/7}$. Furthermore, $\mathbf{r}'(t) = \langle 1,3 \rangle$ so that $|\mathbf{r}'(t)| = \sqrt{1^2 + 3^2} = \sqrt{10}$. The integral is

$$\begin{aligned}
\int_C \left(10 - \frac{1}{8}x^2 - \frac{1}{12}y^2\right) ds &= \int_0^{\sqrt{80/7}} \left(10 - \frac{1}{8}(t)^2 - \frac{1}{12}(3t)^2\right) \sqrt{10}\, dt \\
&= \sqrt{10} \int_0^{\sqrt{80/7}} \left(10 - \frac{7}{8}t^2\right) dt \\
&= \sqrt{10}\left[10t - \frac{7}{24}t^3\right]_0^{\sqrt{80/7}} \\
&= \sqrt{10}\left(10\left(\sqrt{80/7}\right) - \frac{7}{24}\left(\sqrt{80/7}\right)^3\right).
\end{aligned}$$

Using a calculator, the area of the wall is about 71.27 meters^2.

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Example 44.5: Find Find $\int_C (16 - x^2 - y^2)\, ds$, where C is the line from (0,0) to (4,0).

Solution: Path C is parametrized:

$$\mathbf{r}(t) = \langle t, 0 \rangle, \quad \text{for} \quad 0 \le t \le 4.$$

We have $\mathbf{r}'(t) = \langle 1,0 \rangle$ and $|\mathbf{r}'(t)| = \sqrt{1^2 + (0)^2} = 1$. Thus,

$$\int_C (16 - x^2 - y^2)\, ds = \int_0^4 (16 - t^2)\, \underbrace{1\, dt}_{ds} = \frac{128}{3}$$

This example illustrates that the single-variable integrals along the *x*-axis are a special case of the scalar line integral, where the path is a line and the endpoints lie along the *x*-axis. The same would be true for a single-variable integral along the *y*-axis (*x* and *y* being dummy variables in this context).

Suppose that we parameterized the line C from (0,0) to (4,0) as $\mathbf{r}(t) = \langle 4t, 0 \rangle$ for $0 \le t \le 1$. Thus, we have $\mathbf{r}'(t) = \langle 4,0 \rangle$ and $|\mathbf{r}'(t)| = \sqrt{4^2 + (0)^2} = 4$. The line integral is now

$$\begin{aligned} \int_C (16 - (4t)^2 - (0)^2)\, ds &= \int_0^1 (16 - 16t^2)\, 4\, dt \\ &= 64 \int_0^1 (1 - t^2)\, dt \\ &= 64 \left[t - \frac{1}{3} t^3 \right]_0^1 = 64 \left(\frac{2}{3} \right) = \frac{128}{3}. \end{aligned}$$

As expected, we get the same result. Roughly speaking, the first parametrization, $\mathbf{r}(t) = \langle t, 0 \rangle$ where $0 \le t \le 4$, had a “speed” of $|\mathbf{r}'(t)| = 1$, while the second parametrization, with $\mathbf{r}(t) = \langle 4t, 0 \rangle$ and $0 \le t \le 1$ (an interval one-fourth the length of the original interval), covered the same path with a “speed” of $|\mathbf{r}'(t)| = 4$.

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Example 44.6: A glass rod of consistent thickness is in the shape of a quarter-circle of radius 3, modeled by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$, where $0 \le t \le \frac{\pi}{2}$. However, it is not uniformly dense. Suppose that its density x centimeters left and right of the origin, and y centimeters above and below the origin, is given by $d(x,y) = x + y + xy$, where d is grams per centimeter. Find the mass of this glass rod.

Solution: This is equivalent to evaluating the scalar line integral

$$\int_C (x + y + xy)\, ds.$$

Note that $\mathbf{r}'(t) = \langle -3\sin t, 3\cos t\rangle$, so that $|\mathbf{r}'(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2} = 3$. From $\mathbf{r}(t) = \langle 3\cos t, 3\sin t\rangle$, we have $x(t) = 3\cos t$ and $y(t) = 3\sin t$. Substitutions are made:

$$\begin{aligned}
\int_C (x + y + xy)\, ds &= \int_0^{\pi/2} \big((3\cos t) + (3\sin t) + (3\cos t)(3\sin t)\big)\ 3\, dt \\
&= 9\int_0^{\pi/2} (\cos t + \sin t + 3\cos t \sin t)\, dt \\
&= 9\left[\sin t - \cos t + \frac{3}{2}\sin^2 t\right]_0^{\pi/2} \\
&= 9\left(\sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{2}\right) + \frac{3}{2}\sin^2\left(\frac{\pi}{2}\right)\right) - 9\left(\sin(0) - \cos(0) + \frac{3}{2}\sin^2(0)\right) \\
&= 9\left(1 + \frac{3}{2}\right) - 9(-1) \\
&= \frac{63}{2} \text{ grams.}
\end{aligned}$$

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45. Vector Line Integrals: Work & Circulation

Let $\mathbf{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle$ represent a vector field in R^3, and let C be a *directed* path in R^3 parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$. The word "directed" means that the path must be traversed in a specified direction.

The **vector line integral** of **F** along C is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where $d\mathbf{r} = \mathbf{r}'(t) = \frac{d}{dt}\mathbf{r}(t)$. A line integral of this form is also defined in R^2, where the vector field is $\mathbf{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ and C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$.

A common "descriptive" way to describe this line integral is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

As the particle moves along the path C, the vector field either "helps" or "hinders" this particle. In order to remove the particle's speed from consideration, the path is segmented into equally-sized sub-segments using the ds segmentation, where $ds = |\mathbf{r}'(t)|\, dt$. This effectively forces the particle to maintain a constant speed, and without loss of generality, we can use the unit tangent vector, $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, to represent the constant speed. Thus, at any position along the path, one of three situations occurs:

- The vector **F** at this position points in the same direction as **T**. That is, **F** and **T** are acute, and $\mathbf{F} \cdot \mathbf{T} > 0$. Vector **F** is "helping" the particle as though it was pushing it from behind.
- The vector **F** at this position points in an opposing direction as **T**. That is, **F** and **T** are obtuse, and $\mathbf{F} \cdot \mathbf{T} < 0$. Vector **F** is hindering the particle's forward movement, as though it were pushing from the front.
- The vector **F** at this position points in an orthogonal direction as **T**, and $\mathbf{F} \cdot \mathbf{T} = 0$. Vector **F** has no effect on the particle's forward movement.

The integral then sums (in the sense of integration) all of the dot products along the path. If the result of the line integral is positive, then the vector field **F** had a net positive effect on the particle's movement. If the line integral is negative, then the vector field **F** had a net negative effect on the particle's movement. If the line integral is 0, then the vector field **F** had a net-zero effect on the particle's movement.

We take the descriptive form of the line integral and make substitutions:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt\,.$$

Note that $|\mathbf{r}'(t)|$ cancels, so we have

$$\int_C \mathbf{F} \cdot \mathbf{r}'(t)\, dt = \int_a^b \mathbf{F} \cdot d\mathbf{r},$$

where $d\mathbf{r}$ is shorthand for $\mathbf{r}'(t)\, dt$, and $a \leq t \leq b$.

The usual process to determine a line integral is the following:

1) Parameterize the path C in variable t. This will give $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$. It will also give the bounds of integration a and b.
2) Find $\mathbf{r}'(t) = \frac{d}{dt}\mathbf{r}(t)$, which will give $\langle x'(t), y'(t), z'(t) \rangle$.
3) Substitute $x(t)$, $y(t)$ and $z(t)$ (from Step 1) into $\mathbf{F}(x, y, z)$. This will give $\mathbf{F}$ in terms of t.
4) Find $\mathbf{F} \cdot d\mathbf{r}$, which will be a function in terms of t.
5) Integrate the result from Step 4 with respect to t and evaluate at the bounds a and b.

Note: the integral below is a common alternative way to express a line integral:

$$\int_C M(x, y, z)\, dx + N(x, y, z)\, dy + P(x, y, z)\, dz.$$

In this form, the expression $\mathbf{F} \cdot d\mathbf{r}$ has been expanded, where $d\mathbf{r}$ is denoted as $\langle dx, dy, dz \rangle$. It's important to remember that this is equivalent to $\int_C \mathbf{F} \cdot d\mathbf{r}$ and is a single integral in variable t.

The integrals

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds, \quad \int_C \mathbf{F} \cdot d\mathbf{r} \quad \text{and} \quad \int_C M(x, y, z)\, dx + N(x, y, z)\, dy + P(x, y, z)\, dz$$

are all equivalent. These line integrals are used to show the **work** done by a vector field on a particle. If the path is a loop, the movement of a particle along the loop is called **circulation**.

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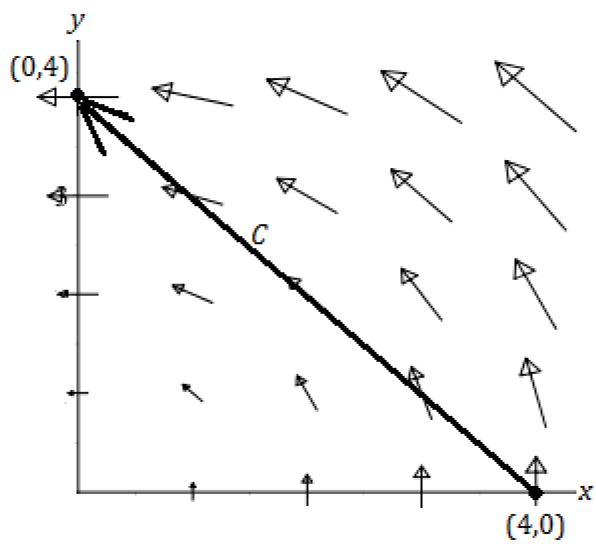
Example 45.1: Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle -y, x \rangle$ and C is the line segment from $P_0 = (4,0)$ to $P_1 = (0,4)$.

Solution: A sketch of the path *C* (in black, with its direction shown by an arrow) with the vectors of **F** show that the vector field generally points in the same direction as the direction of movement along *C*. Thus, we expect that the line integral will be positive.



To find $\int_C \mathbf{F} \cdot d\mathbf{r}$, we follow the steps listed above.

1) Parameterize the path *C* in variable t:

$$\mathbf{r}(t) = \langle 4,0 \rangle + t\langle -4,4 \rangle = \langle 4 - 4t, 4t \rangle, \qquad 0 \leq t \leq 1.$$

2) Find $\mathbf{r}'(t) = \frac{d}{dt}\mathbf{r}(t)$:

$$\frac{d}{dt}\mathbf{r}(t) = \frac{d}{dt}\langle 4 - 4t, 4t \rangle = \langle -4,4 \rangle.$$

3) Substitute $x(t)$ and $y(t)$ (from Step 1) into $\mathbf{F}(x, y)$:

$$\mathbf{F}\big(x(t), y(t)\big) = \langle -4t, 4 - 4t \rangle.$$

4) Find $\mathbf{F} \cdot d\mathbf{r}$:

$$\mathbf{F} \cdot d\mathbf{r} = \langle -4t, 4 - 4t \rangle \cdot \langle -4,4 \rangle = (-4t)(-4) + (4 - 4t)(4) = 16.$$

5) Integrate the result from Step 4 with respect to t:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 16\, dt = 16.$$

The positive quantity of the line integral suggests that particle is "helped" by the vector field as it moves along the path C.

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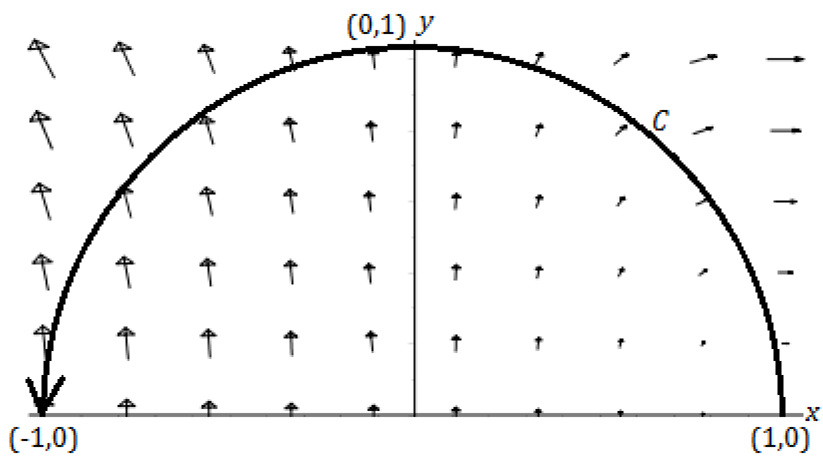
Example 45.2: Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle xy, 1 - x \rangle$ and C is the portion of the unit circle that begins at $P_0 = (1,0)$ and continues counterclockwise to $P_1 = (-1,0)$.

Solution: The path C is sketched and its direction noted. We see that the vectors in $\mathbf{F}$ generally point against (or orthogonal) to the direction of C. Thus, we expect that the line integral will be negative.



Parametrize the path C by the usual parameterization of the unit circle. Note that this is half a circle, so the upper bound of t is π:

$$\mathbf{r}(t) = \langle \cos t, \sin t \rangle, \quad \text{for} \quad 0 \le t \le \pi.$$

As a result, $d\mathbf{r} = \langle -\sin t, \cos t \rangle$ and $\mathbf{F}\big(x(t), y(t)\big) = \langle \cos t \sin t, 1 - \cos t \rangle$. We then have

$$\mathbf{F} \cdot d\mathbf{r} = \langle -\sin t, \cos t \rangle \cdot \langle \cos t \sin t, 1 - \cos t \rangle = -\sin^2 t \cos t + \cos t - \cos^2 t.$$

This is integrated, using *u*-*du* substitution for the first term and an identity for the third term:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^\pi (-\sin^2 t\cos t + \cos t - \cos^2 t)\, dt \\ &= \int_0^\pi \left(-\sin^2 t\cos t + \cos t - \left(\frac{1}{2}+\frac{1}{2}\cos 2t\right)\right) dt \qquad \left\{\cos^2 t = \frac{1}{2}+\frac{1}{2}\cos 2t\right. \\ &= \left[-\frac{1}{3}\sin^3 t + \sin t - \frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^\pi\end{aligned}$$

Evaluated at the bounds, we have

$$\left(-\frac{1}{3}\sin^3\pi + \sin\pi - \frac{1}{2}\pi - \frac{1}{4}\sin 2\pi\right) - \left(-\frac{1}{3}\sin^3 0 + \sin 0 - \frac{1}{2}(0) - \frac{1}{4}\sin 2(0)\right),$$

in which all but one term vanishes. Thus, the result is $-\frac{\pi}{2}$. The particle is “hindered” by the vector field as it moves along the path *C*.

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Example 45.3: Find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \langle x, xy, y+z^2\rangle$ and *C* is the line segment from $P_0 = (1,2,-4)$ to $P_1 = (3,5,1)$.

Solution: The line segment *C* is parameterized as

$$\mathbf{r}(t) = \langle 1+2t, 2+3t, -4+5t\rangle, \qquad 0 \le t \le 1.$$

Now, find $\mathbf{r}'(t) = \frac{d}{dt}\mathbf{r}(t)$:

$$\frac{d}{dt}\mathbf{r}(t) = \frac{d}{dt}\langle 1+2t, 2+3t, -4+5t\rangle = \langle 2,3,5\rangle.$$

Substitute $x(t)$, $y(t)$ and $z(t)$ into $\mathbf{F}(x,y,z)$:

$$\begin{aligned}&\mathbf{F}(x,y,z) = \langle x, xy, y+z^2\rangle \\ &\mathbf{F}(x(t),y(t),z(t)) = \langle 1+2t, (1+2t)(2+3t), (2+3t)+(-4+5t)^2\rangle.\end{aligned}$$

Simplified, we have

$$\mathbf{F}(t) = \mathbf{F}(x(t),y(t),z(t)) = \langle 1+2t, 6t^2+7t+2, 25t^2-37t+18\rangle.$$

Find $\mathbf{F} \cdot d\mathbf{r}$:

$$\begin{aligned}\mathbf{F} \cdot d\mathbf{r} &= \langle 1+2t, 6t^2+7t+2, 25t^2-37t+18\rangle \cdot \langle 2,3,5\rangle \\ &= 2(1+2t)+3(6t^2+7t+2)+5(25t^2-37t+18) \\ &= 143t^2-160t+98.\end{aligned}$$

Now, integrate with respect to t:

$$\begin{aligned}\int_0^1 (143t^2-160t+98)\,dt &= \left[\frac{143}{3}t^3-80t^2+98t\right]_0^1 \\ &= \frac{143}{3}-80+98 \\ &= \frac{197}{3}.\end{aligned}$$

Thus, $\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{197}{3}$, a positive quantity, indicating that the vector field **F** "helped" the particle as it moved from $P_0 = (1,2,-4)$ to $P_1 = (3,5,1)$.

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Example 45.4: Evaluate $\int_C z\,dx + (1-x)\,dy + 2y\,dz$, where C is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$.

Solution: In this form, we can infer the vector field **F** from the information within the integral:

$$\int_C z\,dx + (1-x)\,dy + 2y\,dz.$$

We see that

$$M(x,y,z) = z, \quad N(x,y,z) = 1-x \quad \text{and} \quad P(x,y,z) = 2y,$$

so that $\mathbf{F}(x,y,z) = \langle z, 1-x, 2y\rangle$.

From the helix, we have $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, so that $x(t) = \cos t$, $y(t) = \sin t$ and $z(t) = t$. Thus, $\mathbf{F}(t) = \langle z, 1-x, 2y\rangle = \langle t, 1-\cos t, 2\sin t\rangle$.

Also from the helix, we have $d\mathbf{r} = \frac{d}{dt}\langle \cos t, \sin t, t\rangle = \langle -\sin t, \cos t, 1\rangle$. The line integral is now evaluated:

$$\begin{aligned}\int_C z\,dx + (1-x)\,dy + 2y\,dz &= \int_C \mathbf{F}\cdot d\mathbf{r} \\ &= \int_C \langle t, 1-\cos t, 2\sin t\rangle \cdot \langle -\sin t, \cos t, 1\rangle\,dt \\ &= \int_0^{2\pi} (-t\sin t + (1-\cos t)\cos t + 2\sin t)\,dt \\ &= \int_0^{2\pi} (-t\sin t + \cos t - \cos^2 t + 2\sin t)\,dt\,.\end{aligned}$$

The antiderivative of $-t\sin t$ is found by integration-by-parts and is $t\cos t - \sin t$, while the antiderivative of $-\cos^2 t$ is found by utilizing the identity $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$. When simplified, we have

$$\int_0^{2\pi} (-t\sin t + \cos t - \cos^2 t + 2\sin t)\,dt = \left[t\cos t - \sin t - \frac{1}{2}t - \frac{1}{4}\sin 2t - 2\cos t\right]_0^{2\pi}$$

Evaluated at the bounds, we have $(2\pi - \pi - 2) - (-2) = \pi$.

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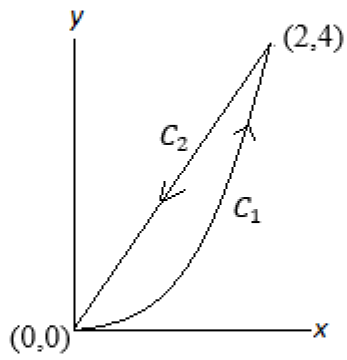
Example 45.5: Evaluate $\int_C xy\,dx + x^2\,dy$, where C is the arc of the parabola $y = x^2$ from (0,0) to (2,4), followed by a straight line from (2,4) back to (0,0).



Solution: From the integral form, we see that $\mathbf{F}(x,y) = \langle xy, x^2\rangle$. The path C is composed of two smaller paths. Let C_1 be the parabolic arc, and C_2 be the line. Thus, the parametrizations are:

$$\begin{aligned}&C_1\!: \ \mathbf{r}_1(t) = \langle t, t^2\rangle, \qquad 0 \le t \le 2,\\ &C_2\!: \ \mathbf{r}_2(t) = \langle 2-2t, 4-4t\rangle, \qquad 0 \le t \le 1.\end{aligned}$$

In such cases, the entire path C is the union of its sub-paths, so that

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_{C_1\cup C_2} \mathbf{F}\cdot d\mathbf{r} \\ &= \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2\,.\end{aligned}$$

For the parabolic arc, we have $d\mathbf{r}_1 = \langle 1{,}2t \rangle$ and $\mathbf{F}(t) = \langle t^3, t^2 \rangle$. Thus,

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 &= \int_{C_1} \langle t^3, t^2 \rangle \cdot \langle 1{,}2t \rangle \, dt \\ &= \int_0^2 3t^3 \, dt \\ &= \left[\frac{3}{4} t^4\right]_0^2 = 12\,.\end{aligned}$$

For the line, we have $d\mathbf{r}_2 = \langle -2, -4 \rangle$ and $\mathbf{F}(t) = \langle 8t^2 - 16t + 8{,}4t^2 - 8t + 4 \rangle$. Thus,

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 &= \int_{C_2} \langle 8t^2 - 16t + 8, 4t^2 - 8t + 4 \rangle \cdot \langle -2, -4 \rangle \, dt \\ &= \int_0^1 (-32t^2 + 64t - 32) \, dt \\ &= \left[-\frac{32}{3} t^3 + 32t^2 - 32t\right]_0^1 = -\frac{32}{3}.\end{aligned}$$

Therefore, $\int_C \; xy \, dx + x^2 \, dy = 12 + \left(-\frac{32}{3}\right) = \frac{4}{3}$.

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