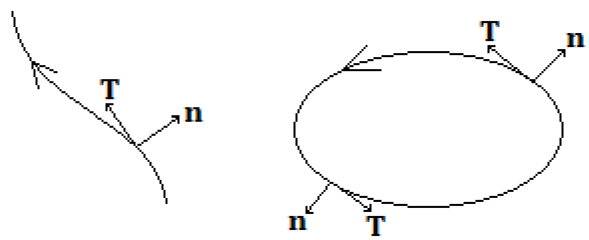
46. Vector Line Integrals: Flux

A second form of a line integral can be defined to describe the flow of a medium through a permeable membrane. Let $\mathbf{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ be a vector field in R^2, representing the flow of the medium, and let *C* be a directed path, representing the permeable membrane. The **flux** (flow) of **F** through *C* is given by the flux line integral

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds.$$

Here, **n** represents a unit normal vector to *C*. The orientation of **n** is important, since it will define the "positive" direction of the flow. There are two cases:

- If *C* is a path with different starting and ending points, then **n** will point orthogonally to **T**, the unit tangent vector, "to the right"; that is, as one moves along *C* in the direction of **T**, then **n** points to the right of **T**. Another way to describe this is that $\mathbf{n}\times\mathbf{T}$ would be in the direction of positive *z*, or out of the sheet of paper toward the viewer.
- If *C* is a simple closed loop (same starting and ending point, does not cross itself) that is traversed counterclockwise, then **n** points outward. Note that this also maintains the "pointing to the right" rule.



Left: Path *C* with a direction defined. Unit tangent **T** points in the forward direction, and **n** is orthogonal to **T** to **T**'s right. **Right:** a simple closed loop path *C* in the counterclockwise orientation. **T** points in the forward direction, and **n** is orthogonal to **T**. Note that in all cases, $\mathbf{n}\times\mathbf{T}$ points up from this page.

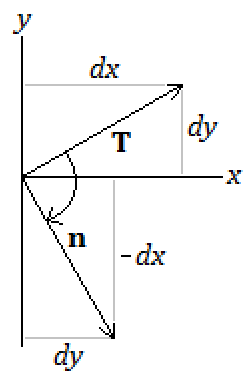
The underlying idea is to compare the direction of **F** to **n** at each point along the path *C*, which is segmented into equally-sized sub-segments for the moment. If **F** and **n** point in the same direction, then their dot product $\mathbf{F}\cdot\mathbf{n}$ is positive. If the two vectors point in opposite directions, then $\mathbf{F}\cdot\mathbf{n}$ is negative, and if the two vectors are orthogonal to one another, then $\mathbf{F}\cdot\mathbf{n}$ is zero. The integral then "sums" all such possible dot products, resulting in a value that represents positive net flow (if the value is positive), negative net flow (if the value is negative), or no net flow (if the value is zero).

If C is parameterized as $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, then recall that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle x'(t), y'(t) \rangle}{\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}}.$$

Since **n** is a right-angle turn clockwise from **T**, then

$$\mathbf{n}(t) = \frac{\langle y'(t), -x'(t) \rangle}{\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}}.$$



If $\mathbf{T} = \langle dx, dy \rangle$, then one 90-degree clockwise turn gives $\mathbf{n} = \langle dy, -dx \rangle$.

Thus, the flux line integral can be rewritten in terms of t:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \mathbf{F}\big(x(t), y(t)\big) \cdot \frac{\langle y'(t), -x'(t) \rangle}{\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

The radicals cancel, and after taking the dot product, we have the short form

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \left(M \frac{dy}{dt} - N \frac{dx}{dt}\right) dt = \int_a^b M\, dy - N\, dx.$$

Remember, despite the notation, this is an integral in variable t.

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Example 46.1: Find the flux of $\mathbf{F}(x,y) = \langle 2,0\rangle$ through the line segment from (3,0) to (0,3).

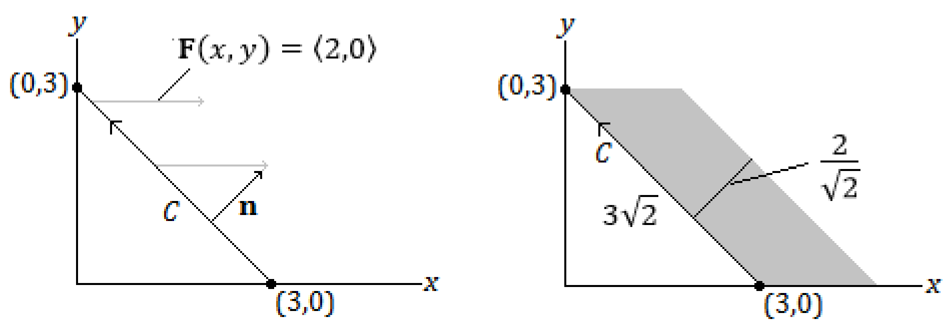
Solution: The line segment *C* is parameterized first:

$$\mathbf{r}(t) = \langle 3-3t, 3t\rangle \quad \text{for} \quad 0 \le t \le 1. \qquad \text{(See Example 12.2)}$$

Thus, $\mathbf{r}'(t) = \langle -3,3\rangle$. From this, we have $\frac{dx}{dt} = -3$ and $\frac{dy}{dt} = 3$. Since **F** is a constant vector field, no substitutions need to be made. The flux of $\mathbf{F}(x,y) = \langle M,N\rangle = \langle 2,0\rangle$ through *C* is given by

$$\int_a^b M\,dy - N\,dx = \int_0^1 \big((2)(3) - (0)(-3)\big)\,dt = \int_0^1 6\,dt = 6.$$

In one unit of time, 6 units of mass flow through *C*. This can be seen graphically as the area of the shaded region below:



Left: Path *C* is shown, along with some vectors $\mathbf{F}(x,y) = \langle 2,0\rangle$ in gray. Each of these vectors has a magnitude of 2 units. Note that the vectors **F** cross *C* at a 45-degree angle, so that the component of these vectors in the direction of **n** is $2/\sqrt{2}$. **Right:** The total flow is represented by the gray parallelogram, whose area is the base (length of *C*) multiplied by the height (the component of **F** in the direction of **n**). We have $(3\sqrt{2})(2/\sqrt{2}) = 6$.

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Example 46.2: Find the flux of $\mathbf{F}(x,y) = \langle 3xy, x-y\rangle$ through the parabolic arc $y = x^2$ between $(-1,1)$ and $(4,16)$.

Solution: The parabolic arc is parameterized as

$$\mathbf{r}(t) = \langle t, t^2\rangle, \quad \text{for} \quad -1 \le t \le 4.$$

Thus, $\mathbf{r}'(t) = \langle 1, 2t\rangle$. Furthermore,

$$\mathbf{F}(t) = \mathbf{F}\big(x(t), y(t)\big) = \langle 3(t)(t^2), (t) - (t^2)\rangle = \langle 3t^3, t - t^2\rangle,$$

where $3xy = 3(t)(t^2) = 3t^3$ and $x - y = (t) - (t^2) = t - t^2$.

Therefore, the flux of $\mathbf{F}(x, y) = \langle 3xy, x - y \rangle$ through the parabolic arc is

$$\int_{-1}^{4} (3t^3)(2t) - (t - t^2)(1)\, dt = \int_{-1}^{4} (6t^4 + t^2 - t)\, dt = \left[\frac{6}{5}t^5 + \frac{1}{3}t^3 - \frac{1}{2}t^2\right]_{-1}^{4}$$

This simplifies to

$$= \left(\frac{6}{5}(4)^5 + \frac{1}{3}(4)^3 - \frac{1}{2}(4)^2\right) - \left(\frac{6}{5}(-1)^5 + \frac{1}{3}(-1)^3 - \frac{1}{2}(-1)^2\right) = \frac{7465}{6}.$$

About 1,244.167 units of mass flow through this membrane per unit of time.

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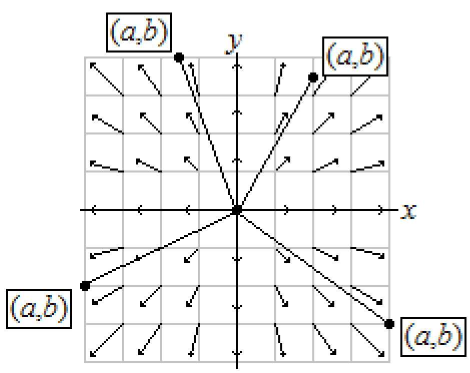
Example 46.3: Find the flux of $\mathbf{F}(x, y) = \langle x, y \rangle$ through the line connecting (0,0) to (*a*,*b*).

Solution: The line is parameterized as $\mathbf{r}(t) = \langle at, bt \rangle$ for $0 \le t \le 1$, and so $\mathbf{r}'(t) = \langle a, b \rangle$. Furthermore, $\mathbf{F}(t) = \mathbf{F}\big(x(t), y(t)\big) = \langle at, bt \rangle$. Thus, the flux is

$$\left.\begin{matrix} M = at & dy = b \\ N = bt & dx = a \end{matrix}\right\} \qquad \int_0^1 (at)(b) - (bt)(a)\, dt = \int_0^1 (abt - abt)\, dt = 0.$$

This result is not surprising: any line connecting (0,0) to (*a*,*b*) is parallel to the stream-lines formed by the vector field $\mathbf{F}(x, y) = \langle x, y \rangle$. At no time (or place) does **F** ever pass through such a line.



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Example 46.4: Find the flux of $\mathbf{F}(x,y) = \langle 3x, 5y\rangle$ through the circle $x^2 + y^2 = 1$, traversed counterclockwise.

Solution: The circle is parameterized as $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le 2\pi$. Thus, $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$. The vector field is written in terms of t:

$$\mathbf{F}(t) = \mathbf{F}(x(t), y(t)) = \langle 3\cos t, 5\sin t\rangle.$$

Thus, we have $M = 3\cos t$, $N = 5\sin t$, $\frac{dy}{dt} = \cos t$ and $\frac{dx}{dt} = -\sin t$:

$$\int_0^{2\pi} \big((3\cos t)(\cos t) - (5\sin t)(-\sin t)\big)\, dt = \int_0^{2\pi} (3\cos^2 t + 5\sin^2 t)\, dt.$$

We use the identities $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos 2t$ and $\sin^2 t = \frac{1}{2} - \frac{1}{2}\cos 2t$ to simplify the integrand:

$$\begin{aligned}\int_0^{2\pi} (3\cos^2 t + 5\sin^2 t)\, dt &= \int_0^{2\pi} \left(3\left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) + 5\left(\frac{1}{2} - \frac{1}{2}\cos 2t\right)\right) dt \\ &= \int_0^{2\pi} (4 - \cos 2t)\, dt \\ &= \left[4t - \frac{1}{2}\sin 2t\right]_0^{2\pi} = 8\pi.\end{aligned}$$

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The flux of $\mathbf{F} = \langle M, N\rangle$ through a simple closed loop C traversed counterclockwise, such as in Example 46.4, can be calculated using the following formula:

$$\int_C \mathbf{F}\cdot\mathbf{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA,$$

where R is the region enclosed by C. This is called the **Divergence Theorem** in R^2. The general Divergence Theorem is discussed later in Section 54.

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Example 46.5: Use the Divergence Theorem to find the flux of $\mathbf{F}(x,y) = \langle 5y^2, \sqrt{e^x}\rangle$ through the triangle traversed from vertices (1,1), (5,1) and (3,5), back to (1,1), in that order.

Solution: Note that $\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0$. Thus, the net flux is 0. This means that equal amounts of mass are entering and exiting through the boundaries per unit of time.

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Example 46.6: Use the Divergence Theorem to find the flux of $\mathbf{F}(x, y) = \langle 3x, 5y \rangle$ through the circle $x^2 + y^2 = 1$, traversed counterclockwise. (This is a repeat of Example 46.4)

Solution: Since $M(x, y) = 3x$, then $\frac{\partial M}{\partial x} = 3$, and since $N(x, y) = 5y$, then $\frac{\partial N}{\partial y} = 5$. Thus, we have

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \iint_R (3 + 5)\, dA = \iint_R 8\, dA = 8 \iint_R dA.$$

Note that $\iint_R dA$ represents the area of *R*. Since *R* is a circle of radius 1, its area is $\pi(1)^2 = \pi$. Thus, the flux of $\mathbf{F}(x, y) = \langle 3x, 5y \rangle$ through the circle $x^2 + y^2 = 1$ is

$$8 \iint_R dA = 8\pi.$$

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Example 46.7: Use the Divergence Theorem to find the flux of $\mathbf{F}(x, y) = \langle xy, 3 \rangle$ through the rectangle traversed from (0,0) to (0,3) to (6,3) to (6,0) to (0,0).

Solution: The path *C* is a rectangle traced clockwise, not counterclockwise as is required by the Divergence Theorem. We can proceed, but must negate the final result to account for the clockwise movement. We have

$$\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA = \int_0^6 \int_0^3 y\, dy\, dx.$$

The inside integral is

$$\int_0^3 y\, dy = \left[\frac{1}{2}y^2\right]_0^3 = \frac{9}{2}.$$

The outside integral is

$$\int_0^6 \left(\frac{9}{2}\right) dx = \left(\frac{9}{2}\right)(6) = 27.$$

Since the path around the rectangle is traced clockwise, we negate the result:

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = -27.$$

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