32. Method of Lagrange Multipliers

The Method of Lagrange Multipliers is a generalized approach to solving constrained optimization problems. Assume that we are seeking to optimize a function $z = f(x, y)$ subject to a "path" constraint defined implicitly by $g(x, y) = c$. The process usually follows these steps:

1. Define a function $L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$.
2. Find the partial derivatives L_x, L_y and L_λ. Note that $L_\lambda = -g(x, y) + c$.
3. Set these partial derivatives to 0. Note that $L_\lambda = 0$ is the same as $g(x, y) = c$, the original path constraint. Also note any restrictions on *x* and *y*, as it may be necessary to consider locations where the derivative fails to exist.
4. Isolate the λ in the equations $L_x = 0$ and $L_y = 0$, then equate the two expressions. This will "drop out" the λ, leaving an equation in *x* and *y* only. If possible, isolate *x* or *y*.
5. Substitute the result from step 4 into the equation $g(x, y) = c$, which will now be a single-variable equation. Solve for the remaining variable.
6. Back substitute to find corresponding values for the other variable, and for *z*.
7. Compare *z* values. The smallest will be a minimum, the largest a maximum. If there is just one *z* value, then other observations, such as cross-sections, may be needed to determine whether the point is a minimum or maximum.

The following examples illustrate possible situations that may occur.

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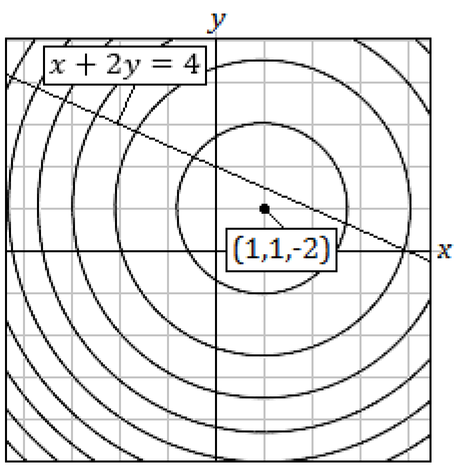
Example 32.1: Find the minimum value of $z = f(x, y) = x^2 + y^2 - 2x - 2y$ subject to the constraint $x + 2y = 4$.

Solution: To the right is a contour map in R^2 of the surface defined by f, and the constraint $x + 2y = 4$ shown as a line. The actual surface is a paraboloid that opens up and has a minimum point at $(1, 1, -2)$, its vertex. The path, when conformed to the surface, is a cross-section of the paraboloid, itself a parabola. Thus, by inspecting the geometry of the problem, the extreme point on this path/parabola will be a minimum point.



Note that the path is slightly off-set from the vertex. It is reasonable to assume that the lowest point on the constraint path will be near the vertex, but clearly cannot be at the paraboloid's vertex.

First, create a new function L, clearing parentheses at the end:

$$L(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - c)$$
$$L(x, y, \lambda) = x^2 + y^2 - 2x - 2y - \lambda(x + 2y - 4)$$
$$L(x, y, \lambda) = x^2 + y^2 - 2x - 2y - \lambda x - 2\lambda y + 4\lambda.$$

Next, find its partial derivatives:

$$L_x = 2x - 2 - \lambda$$
$$L_y = 2y - 2 - 2\lambda$$
$$L_\lambda = -x - 2y + 4.$$

Now, set these to 0:

$$2x - 2 - \lambda = 0 \quad (\mathbf{1})$$
$$2y - 2 - 2\lambda = 0 \quad (\mathbf{2})$$
$$-x - 2y + 4 = 0. \quad (\mathbf{3})$$

In equations **(1)** and **(2)**, isolate the λ:

$$\lambda = 2x - 2 \quad \text{and} \quad \lambda = y - 1.$$

There are no restrictions on *x* or *y*. Now, equate and simplify. Note that λ is no longer present.

$$2x - 2 = y - 1, \quad \text{which gives} \quad y = 2x - 1.$$

Note that equation **(3)** from above is the same as the constraint $x + 2y = 4$. Substitute the equation $y = 2x - 1$ into the simplified form of equation **(3)**, and solve for *x*:

$$x + 2(2x - 1) = 4$$
$$5x - 2 = 4$$
$$5x = 6$$
$$x = \frac{6}{5}.$$

Find y by substituting $x = \frac{6}{5}$ into the equation $y = 2x - 1$:

$$y = 2\left(\frac{6}{5}\right) - 1 = \frac{7}{5}\,.$$

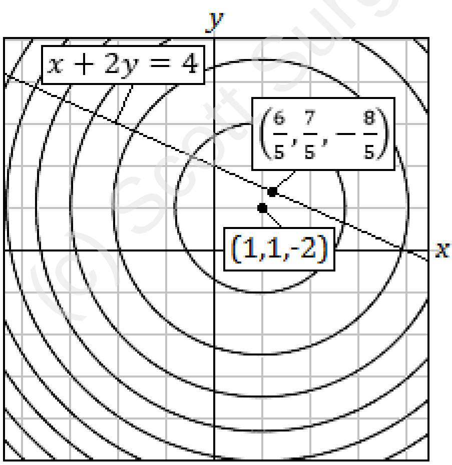
Lastly, find z using the original function f:

$$f\left(\frac{6}{5}, \frac{7}{5}\right) = \left(\frac{6}{5}\right)^2 + \left(\frac{7}{5}\right)^2 - 2\left(\frac{6}{5}\right) - 2\left(\frac{7}{5}\right) = -\frac{8}{5}\,.$$

The minimum point of $z = f(x, y) = x^2 + y^2 - 2x - 2y$ subject to the constraint $x + 2y = 4$ is

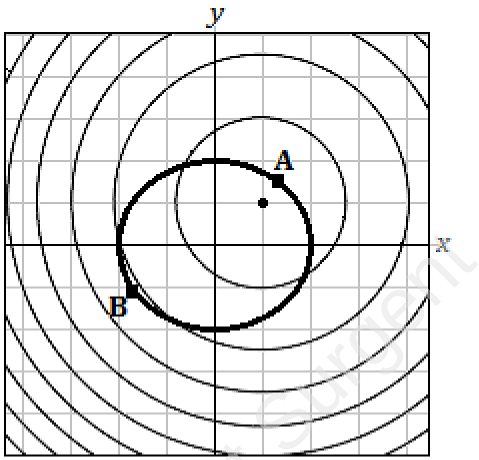
$$\left(\frac{6}{5}, \frac{7}{5}, -\frac{8}{5}\right).$$

This seems to agree with our assumption that it would be "close" to the surface's minimum at $(1,1,-2)$, its component values each a little higher than those of the vertex.



Example 32.2: Let $z = f(x, y) = x^2 + y^2 - 2x - 2y$. Find the minimum value of f subject to the constraint $x^2 + y^2 = 4$.

Solution: This is the same surface as in the previous example. However, the constraint path is a circle of radius 2 (as viewed on the *xy*-plane). When conformed to the surface f, the path will rise and fall along with the surface. Observing the path (in bold) in relation to the contours, we can estimate where the path's lowest point may be, and where its highest point may be:



A: Probable lowest point
B: Probable highest point

Using the Method of Lagrange Multipliers, we start by building function L:

$$L(x, y, \lambda) = x^2 + y^2 - 2x - 2y - \lambda(x^2 + y^2 - 4)$$
$$L(x, y, \lambda) = x^2 + y^2 - 2x - 2y - \lambda x^2 - \lambda y^2 + 4\lambda. \quad \text{(Simplified)}$$

Now find the partial derivatives:

$$\begin{aligned} L_x &= 2x - 2 - 2\lambda x \\ L_y &= 2y - 2 - 2\lambda y \\ L_\lambda &= -x^2 - y^2 + 4. \end{aligned}$$

Set each partial derivative to 0. Again, note that $L_\lambda = 0$ (Equation **(3)**) is the constraint path:

$$\begin{aligned} 2x - 2 - 2\lambda x &= 0 \quad (\mathbf{1}) \\ 2y - 2 - 2\lambda y &= 0 \quad (\mathbf{2}) \\ x^2 + y^2 &= 4. \quad (\mathbf{3}) \end{aligned}$$

Isolate λ in equations **(1)** and **(2)**, then equate. Note any restrictions on the variables:

$$\lambda = \frac{x-1}{x} \quad \text{and} \quad \lambda = \frac{y-1}{y}, \quad \text{so that} \quad \frac{x-1}{x} = \frac{y-1}{y} \quad (x, y \neq 0).$$

Clearing fractions, we have

$$\begin{aligned} y(x-1) &= x(y-1) \\ xy - y &= xy - x \\ y &= x. \end{aligned}$$

Substitute $y = x$ into **(3)**:

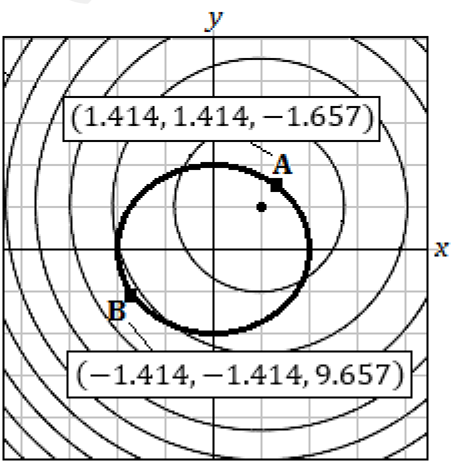
$$\begin{aligned} x^2 + x^2 &= 4 \\ 2x^2 &= 4 \\ x^2 &= 2 \\ x &= \pm\sqrt{2}. \end{aligned}$$

Since $y = x$, we have $y = \sqrt{2}$ when $x = \sqrt{2}$, and $y = -\sqrt{2}$ when $x = -\sqrt{2}$. There are two critical points:

$$\left(\sqrt{2}, \sqrt{2}, f\left(\sqrt{2}, \sqrt{2}\right)\right) \quad \text{and} \quad \left(-\sqrt{2}, -\sqrt{2}, f\left(-\sqrt{2}, -\sqrt{2}\right)\right).$$

We now evaluate the function at each of these *x* and *y* values:

$$\begin{aligned} f\left(\sqrt{2}, \sqrt{2}\right) &= \left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 - 2\sqrt{2} - 2\sqrt{2} = 4 - 4\sqrt{2} \approx -1.657, \\ f\left(-\sqrt{2}, -\sqrt{2}\right) &= \left(-\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2 + 2\sqrt{2} + 2\sqrt{2} = 4 + 4\sqrt{2} \approx 9.657. \end{aligned}$$



By observation,

$$\left(\sqrt{2},\sqrt{2},f\left(\sqrt{2},\sqrt{2}\right)\right)\approx(1.414,1.414,-1.657)$$

is the minimum point (**A**) on the surface subject to the constraint, while

$$\left(-\sqrt{2},-\sqrt{2},f\left(-\sqrt{2},-\sqrt{2}\right)\right)\approx(-1.414,-1.414,9.657)$$

is the maximum point (**B**) on the surface subject to the constraint. We also see that these points are where we surmised they would be: the minimum point on the path is closest to the minimum point of the entire surface, while the maximum point is farthest away.

The restrictions that $x\neq 0$ or $y\neq 0$ ultimately did not play a role in this example. In the next example, it does.

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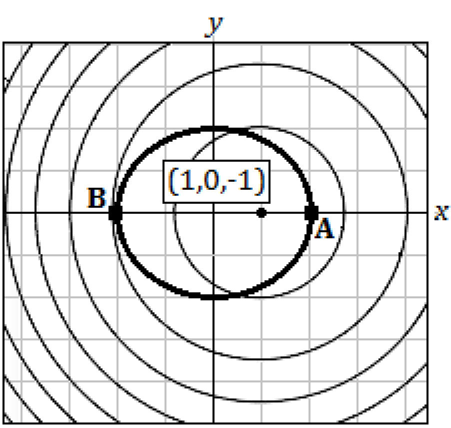
Example 32.3: Let $z=f(x,y)=x^2+y^2-2x$. Find the minimum and maximum values of f subject to the constraint $x^2+y^2=4$.

Solution: The surface as defined by f is a paraboloid with vertex at $(1,0,-1)$. Since the paraboloid opens upward, the vertex is the absolute minimum point on the surface. We show the contour map and identify the path constraint (in bold), which is the circle of radius 2, centered at the origin. It is reasonable to infer that the minimum point on the surface subject to the constraint is probably the point closest to the vertex (denoted **A**), and the maximum point is farthest away from the vertex (denoted **B**) given that the surface rises the farther away one moves from the origin.



We build function L:

$$L(x, y, \lambda) = x^2 + y^2 - 2x - \lambda x^2 - \lambda y^2 + 4\lambda.$$

Now find the partial derivatives:

$$\begin{aligned} L_x &= 2x - 2 - 2\lambda x \\ L_y &= 2y - 2\lambda y \\ L_\lambda &= -x^2 - y^2 + 4. \end{aligned}$$

Set each partial derivative to 0:

$$\begin{aligned} 2x - 2 - 2\lambda x &= 0 \quad (\mathbf{1}) \\ 2y - 2\lambda y &= 0 \quad (\mathbf{2}) \\ x^2 + y^2 &= 4. \quad (\mathbf{3}) \end{aligned}$$

Isolate λ in equations **(1)** and **(2)**, then equate. Note any restrictions on the variables:

$$\lambda = \frac{x-1}{x} \quad \text{and} \quad \lambda = \frac{y}{y} = 1, \quad \text{so that} \quad \frac{x-1}{x} = 1 \quad (x, y \neq 0)$$

Simplifying $\frac{x-1}{x} = 1$, we get $x - 1 = x$, or $0 = -1$, which is a false statement. It seems the process has stalled. However, it has not. The nature of the algebra in this step forces $x \neq 0$ and $y \neq 0$, but in truth, the surface and the constraint are defined when $x = 0$ or $y = 0$. In equation **(2)**, which is $2y - 2\lambda y = 0$, note that $y = 0$ is also a solution.

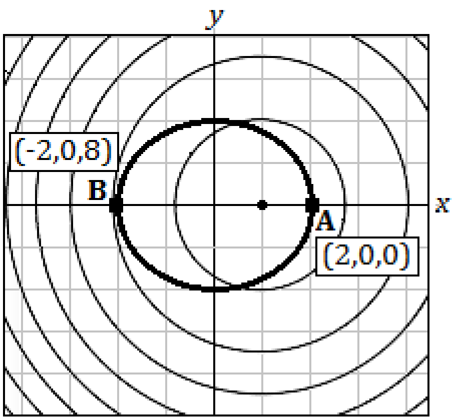
Substituting this into equation **(3)**, the original constraint, we can solve for x:

$$\begin{aligned} x^2 + 0^2 &= 4 \\ x^2 &= 4 \\ x &= \pm 2. \end{aligned}$$

Thus,we have two critical points, $\big(2,0,f(2,0)\big)$ and $\big(-2,0,f(-2,0)\big)$. The z values are

$$f(2,0) = 2^2 + 0^2 - 2(2) = 0 \quad \& \quad f(-2,0) = (-2)^2 + 0^2 - 2(-2) = 8.$$

The point $(2,0,0)$ is the minimum point (**A**) on the surface subject to the constraint, and the point $(-2,0,8)$ is the maximum point (**B**) on the surface subject to the constraint. This agrees with our original intuition.



Graph for example 32.3

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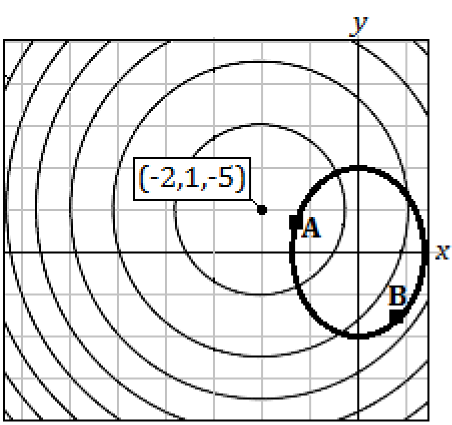
The previous three examples have been efficient, in that the algebra has not been too difficult. In the next example, we encounter a situation where the algebra may pose a challenge.

Example 32.4: Let $z = f(x,y) = x^2 + y^2 + 4x - 2y$. Find the minimum and maximum values of f subject to the constraint $2x^2 + y^2 = 4$.

Solution: The surface is a paraboloid opening upward. Its vertex, $(-2,1,-5)$, is the absolute minimum point on this surface. The path is an ellipse centered at the origin with a major axis of 4 units in the y direction (± 2 units from the origin) and a minor axis of $2\sqrt{2}$ units in the x direction ($\pm\sqrt{2}$ units from the origin). We label what we think may be the location of the minimum point (**A**) of the surface subject to the constrain, and what we think may be the maximum point (**B**) of the surface, subject to the constraint.



We follow the same steps as before:

$$L(x, y, \lambda) = x^2 + y^2 + 4x - 2y - 2\lambda x^2 - \lambda y^2 + 4\lambda.$$

The partial derivatives are

$$\begin{aligned} L_x &= 2x + 4 - 4\lambda x \\ L_y &= 2y - 2 - 2\lambda y \\ L_\lambda &= -2x^2 - y^2 + 4. \end{aligned}$$

Setting each to 0, we have a system:

$$\begin{aligned} 2x + 4 - 4\lambda x &= 0 \quad (\mathbf{1}) \\ 2y - 2 - 2\lambda y &= 0 \quad (\mathbf{2}) \\ 2x^2 + y^2 &= 4. \quad (\mathbf{3}) \end{aligned}$$

Isolate λ in equations **(1)** and **(2)**, then equate. Note any restrictions on the variables:

$$\lambda = \frac{x+2}{2x} \quad \text{and} \quad \lambda = \frac{y-1}{y}, \quad \text{so that} \quad \frac{x+2}{2x} = \frac{y-1}{y} \quad (x, y \neq 0)$$

Clearing fractions, we have

$$\begin{aligned} y(x+2) &= 2x(y-1) \\ xy + 2y &= 2xy - 2x \\ 2y - xy &= -2x \\ y(2-x) &= -2x \\ y &= \frac{2x}{x-2} \qquad (x \neq 2). \end{aligned}$$

Substitute this into **(3)**:

$$2x^2 + \left(\frac{2x}{x-2}\right)^2 = 4.$$

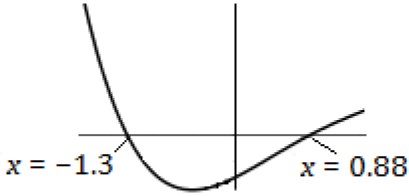
Clear fractions:

$$(x-2)^2 2x^2 + (2x)^2 = 4(x-2)^2.$$

Expanding by multiplication and collecting terms, we have

$$x^4 - 4x^3 + 4x^2 + 8x - 8 = 0.$$

It is difficult to isolate x in a quartic polynomial. Instead, the roots are determined graphically. The roots are $x \approx -1.3$ and $x \approx 0.88$:



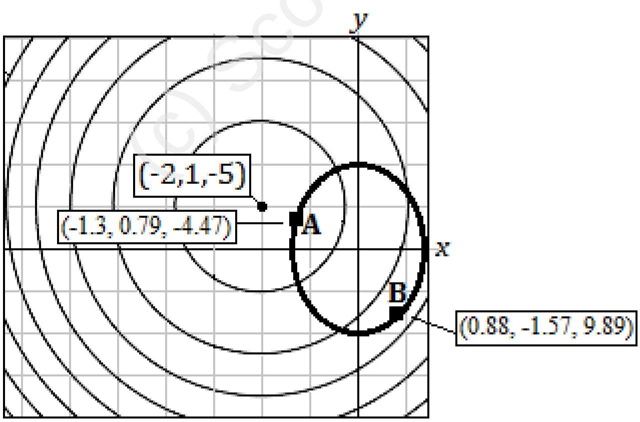
Now use the equation $y = \frac{2x}{x-2}$ to determine y at each x-value:

$$y = \frac{2(-1.3)}{(-1.3)-2} \approx 0.79 \quad \text{and} \quad y = \frac{2(0.88)}{(0.88)-2} \approx -1.57.$$

We then find the z-values:

$$z = f(-1.3, 0.79) = (-1.3)^2 + (0.79)^2 + 4(-1.3) - 2(0.79) \approx -4.47,$$
$$z = f(0.88, -1.57) = (0.88)^2 + (-1.57)^2 + 4(0.88) - 2(-1.57) \approx 9.89.$$

Thus, the point $(-1.3, 0.79, -4.47)$ is the minimum point (**A**) on the surface subject to the constraint, and the point $(0.88, -1.57, 9.89)$ is the maximum point (**B**) on the surface subject to the constraint.



The restrictions imposed on x and y during the algebra steps did not play a role in finding the solutions.

Lagrange Multiplies can be extended into situations with three or more variables.

Example 32.5: Consider the portion of the plane $2x + 4y + 5z = 20$ in the first octant. Find the point on the plane closest to the origin. (This is the same as Example 30.4)

Solution: If (x, y, z) is a point on the plane, then its distance from the origin is $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. Using the constraint $2x + 4y + 5z - 20 = 0$, we build the function L:

$$L(x, y, z, \lambda) = \sqrt{x^2 + y^2 + z^2} - \lambda(2x + 4y + 5z - 20).$$

Then we find partial derivatives:

$$\begin{aligned} L_x &= \frac{x}{\sqrt{x^2 + y^2 + z^2}} - 2\lambda, \\ L_y &= \frac{y}{\sqrt{x^2 + y^2 + z^2}} - 4\lambda, \\ L_z &= \frac{z}{\sqrt{x^2 + y^2 + z^2}} - 5\lambda, \\ L_\lambda &= -2x - 4y - 5z + 20. \end{aligned}$$

These are then set equal to 0:

$$\frac{x}{\sqrt{x^2 + y^2 + z^2}} - 2\lambda = 0, \quad \text{so that} \quad \lambda = \frac{x}{2\sqrt{x^2 + y^2 + z^2}}, \qquad (\mathbf{1})$$

$$\frac{y}{\sqrt{x^2 + y^2 + z^2}} - 4\lambda = 0, \quad \text{so that} \quad \lambda = \frac{y}{4\sqrt{x^2 + y^2 + z^2}}, \qquad (\mathbf{2})$$

$$\frac{z}{\sqrt{x^2 + y^2 + z^2}} - 5\lambda = 0, \quad \text{so that} \quad \lambda = \frac{z}{5\sqrt{x^2 + y^2 + z^2}}, \qquad (\mathbf{3})$$

$$-2x - 4y - 5z + 20 = 0, \quad \text{so that} \quad 2x + 4y + 5z = 20. \qquad (\mathbf{4})$$

Equating **(1)** and **(2)**, we have

$$\frac{x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{y}{4\sqrt{x^2 + y^2 + z^2}}.$$

Clearing fractions, we have $y = 2x$. Then, equating **(1)** and **(3)** and clearing fractions, we have $z = \frac{5}{2}x$. These are substituted into equation **(4)**:

$$2x + 4(2x) + 5\left(\frac{5}{2}x\right) = 20.$$

Simplifying, we have $\frac{45}{2}x = 20$, so that $x = \frac{40}{45} = \frac{8}{9}$. Since $y = 2x$, we have $y = 2\left(\frac{8}{9}\right) = \frac{16}{9}$, and since $z = \frac{5}{2}x$, we have $z = \frac{5}{2}\left(\frac{8}{9}\right) = \frac{40}{18} = \frac{20}{9}$.

The point on the plane $2x + 4y + 5z = 20$ closest to the origin is $\left(\frac{8}{9}, \frac{16}{9}, \frac{20}{9}\right)$.

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Example 32.6: Consider the portion of the plane $2x + 4y + 5z = 20$ in the first octant. A rectangular box is situated with one corner at the origin and its opposite corner on the plane so that the box's edges lie along (or are parallel to) the *x*-axis, *y*-axis or *z*-axis. Find the largest possible volume of such a box, keeping the box to within the first octant. (This is the same as Example 30.5)

Solution: The volume of the box is given by $V(x, y, z) = xyz$, and along with the constraint $2x + 4y + 5z - 20 = 0$, we build function L:

$$L(x, y, z, \lambda) = xyz - \lambda(2x + 4y + 5z - 20).$$

Taking partial derivatives, we have

$$\begin{aligned} L_x &= yz - 2\lambda, \\ L_y &= xz - 4\lambda, \\ L_z &= xy - 5\lambda, \\ L_\lambda &= -2x - 4y - 5z + 20. \end{aligned}$$

Setting each to 0, we have

$$yz - 2\lambda = 0, \quad \text{so that} \quad \lambda = \frac{yz}{2}, \quad (\mathbf{1})$$

$$xz - 4\lambda = 0, \quad \text{so that} \quad \lambda = \frac{xz}{4}, \quad (\mathbf{2})$$

$$xy - 5\lambda = 0, \quad \text{so that} \quad \lambda = \frac{xy}{5}, \quad (\mathbf{3})$$

$$-2x - 4y - 5z + 20 = 0, \quad \text{so that} \quad 2x + 4y + 5z = 20. \quad (\mathbf{4})$$

Equating **(1)** and **(2)**, we have

$$\frac{yz}{2} = \frac{xz}{4}, \quad \text{so that} \quad y = \frac{1}{2}x.$$

Equating **(1)** and **(3)**, we have

$$\frac{yz}{2} = \frac{xy}{5}, \quad \text{so that} \quad z = \frac{2}{5}x.$$

These are substituted into **(4)** and variable *x* is isolated:

$$\begin{aligned} 2x + 4\left(\frac{1}{2}x\right) + 5\left(\frac{2}{5}x\right) &= 20 \\ 2x + 2x + 2x &= 20 \\ 6x &= 20 \\ x &= \frac{10}{3}. \end{aligned}$$

Since $y = \frac{1}{2}x$, we have $y = \frac{1}{2}\left(\frac{10}{3}\right) = \frac{5}{3}$, and since $z = \frac{2}{5}x$, we have $z = \frac{2}{5}\left(\frac{10}{3}\right) = \frac{4}{3}$. Thus, the dimensions of the largest box will be $\frac{10}{3} \times \frac{5}{3} \times \frac{4}{3}$, with the volume $\left(\frac{10}{3}\right)\left(\frac{5}{3}\right)\left(\frac{4}{3}\right) = \frac{200}{27}$.