Jacobians

If we change the variable in an integral, a number or variable expression usually "remains" as part of the new integral. This is called a Jacobian. It can be derived geometrically or analytically.

For example, suppose we have

$$\int x(x^2+1)^4\,dx.$$

If we change the variable of integration by letting $u = x^2 + 1$ and $du = 2x\,dx$, then we have $x\,dx = \frac{1}{2}du$, and we make the variable switch:

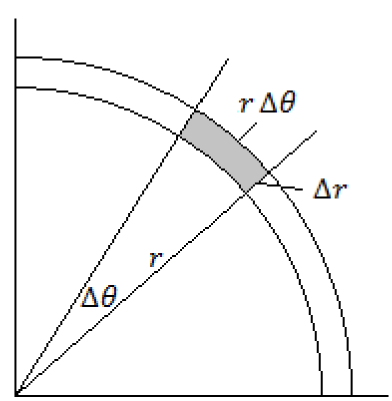
$$\begin{aligned}\int x(x^2+1)^4\,dx &= \int (x^2+1)^4\,x\,dx \\ &= \int (u)^4\left(\tfrac{1}{2}du\right) \qquad \begin{array}{c} x^2+1=u \\ x\,dx=\tfrac{1}{2}du \end{array} \\ &= \tfrac{1}{2}\int u^4\,du.\end{aligned}$$

The $\frac{1}{2}$ that remains as part of the new integration is the Jacobian that is a result of the variable switch.

- ***Polar Coordinates***

The typical area element ΔA in polar form looks "sort of" like a rectangle. It is sometimes called a *polar rectangle*.

Moving away from the origin, we have a small change in *r*, or Δr. This is the length of one side of the polar rectangle. Sweeping the angle θ slightly, we have $\Delta\theta$. In general, the length of a circular arc with radius *r* swept out by central angle θ is $r\theta$. Thus, the other side of the polar rectangle has length $r\Delta\theta$.



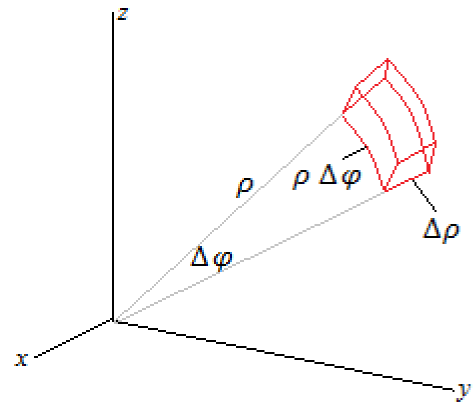
Therefore, the area is the product, $\Delta A = (r\Delta\theta)\Delta r$, or $r\,\Delta r\,\Delta\theta$. Letting the difference tend to zero, we get the standard area differential, $dA = r\,dr\,d\theta$.

- ***Spherical Coordinates***

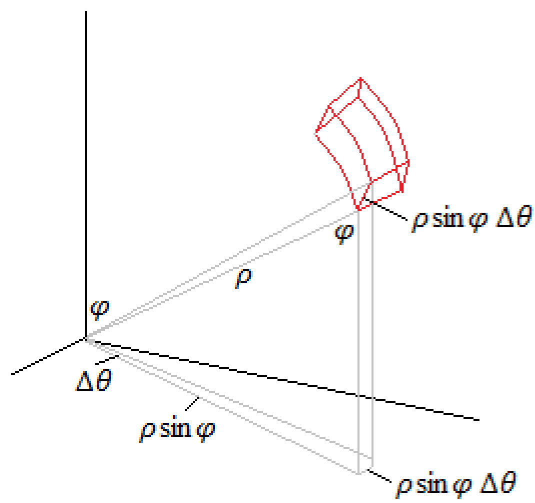
A point in R^3 can be described by its distance ρ (rho) from the origin, and two angles: the "sweep" angle θ which governs its location relative to the *xy*-plane, and the "equilibrium" angle φ (phi), which governs its location relative to the positive *z*-axis.

We look at a typical volume element, which is formed by two spherical sectors (changes in ρ) as well as two changes in the angles θ and φ. If the changes are small enough, we can approximate the volume as though it were a rectangular solid, where $V = lwh$.

In the image at right, the measures of two of the three sides of the rectangle are shown. One side is just the change in spherical radius, $\Delta\rho$, while another side uses the formula for the length of an arc of a circle given its radius and central angle (see the Polar Coordinate discussion earlier). Here, the radius is ρ and the angle is $\Delta\varphi$, so that another side is their product, $\rho\Delta\varphi$.



For the third length, we use more geometry. Since φ is the angle of the "radius line" with length ρ and the positive *z*-axis, then by transversals, φ can also be placed into the corner of a right triangle directly adjacent to one corner of the volume element. Therefore, the opposite side of this triangle has length $\rho \sin\varphi$. Now, this length is swept out on the *xy*-plane by angle $\Delta\theta$, so therefore, the small circular arc on the xy-plane has length $\rho \sin\varphi\ \Delta\theta$. This then is translated up to the actual volume element. Thus, the volume is



$$\Delta V = (\Delta\rho)(\rho\Delta\varphi)(\rho \sin\varphi\ \Delta\theta).$$

As these lengths approach zero, they become the standard Jacobian for a triple integral in spherical coordinates:

$$dV = \rho^2 \sin\varphi\ \ d\rho\ d\theta\ d\varphi.$$