42. Change of Variables: The Jacobian

It is common to change the variable(s) of integration, the main goal being to rewrite a complicated integrand into a simpler equivalent form. However, in doing so, the underlying geometry of the problem may be altered. This is seen often in single-variable integrals:

Example 42.1: Evaluate

$$\int_2^4 (3x+1)^3\,dx.$$

Solution: Let $u = 3x + 1$, so that $du = 3\,dx$. Then substitutions are made. Note that the expression $du = 3\,dx$ is rewritten as $dx = \frac{1}{3}du$; and that the bounds $2 \leq x \leq 4$ are recalculated using $u = 3x + 1$, so that when $x = 2$, then $u = 3(2) + 1 = 7$, and when $x = 4$, then $u = 3(4) + 1 = 13$. The bounds with respect to u are now $7 \leq u \leq 13$.

$$\int_{x=2}^{x=4} (3x+1)^3\,dx = \int_{u=3(2)+1=7}^{u=3(4)+1=13} u^3\left(\frac{1}{3}\,du\right).$$

The integral in terms of u is simplified:

$$\int_{u=3(2)+1=7}^{u=3(4)+1=13} u^3\left(\frac{1}{3}\,du\right) = \frac{1}{3}\int_7^{13} u^3\,du.$$

It is important to note that the two integrals, $\int_2^4 (3x+1)^3\,dx$ and $\frac{1}{3}\int_7^{13} u^3\,du$, represent the identical problem. The $\frac{1}{3}$ that remains after simplification is called the **Jacobian**.

The integral in variable x is over an interval of length 2 units, while the integral in u is over an interval of length 6 units. Roughly speaking, variable u covers its interval of integration (length 6) three times "as fast" as that of x (length 2), and since u and x are linearly related, the leading $\frac{1}{3}$ adjusts for the change in the underlying geometry of the intervals.

For double integrals in R^2, we assume that a region of integration defined in terms of variables *x* and *y* are substituted for new variables *u* and *v* through two functions:

$$u = f_1(x, y)$$
$$v = f_2(x, y).$$

Note that the pair of equations are written so that *u* and *v* are written in terms of *x* and *y*. This is called a **transformation**. Such a "variable change" should be reversible. That is, we should be able to, through algebraic means, isolate *x* and *y* in terms of *u* and *v*:

$$x = g_1(u, v)$$
$$y = g_2(u, v).$$

The **Jacobian** is then defined as a determinant of a 2 by 2 matrix:

$$\mathrm{J}(u, v) = \begin{vmatrix} \partial g_1/\partial u & \partial g_1/\partial v \\ \partial g_2/\partial u & \partial g_2/\partial v \end{vmatrix}, \quad \text{or} \quad \mathrm{J}(u, v) = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}.$$

This can be extended into higher dimensions as well.

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Example 42.2: Find the Jacobian for the transformation

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

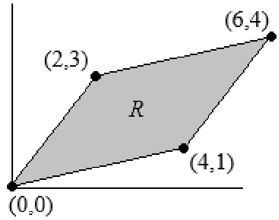
Solution: We have

$$\begin{aligned} \mathrm{J}(r, \theta) &= \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta \\ \partial y/\partial r & \partial y/\partial \theta \end{vmatrix} \\ &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r(\cos^2\theta + \sin^2\theta) \\ &= r. \end{aligned}$$

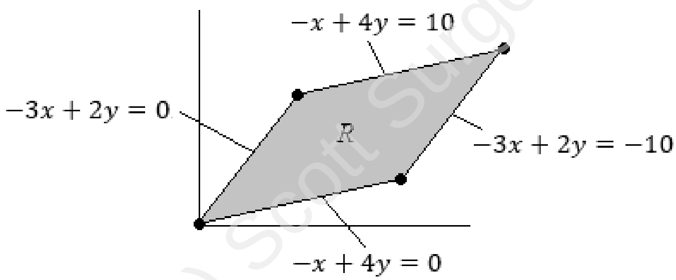
This is the common Jacobian when rectangular coordinates *x* and *y* are rewritten in polar coordinates r and θ.

Example 42.3: Evaluate $\iint_R (x-2y)\,dA$, where R is the parallelogram in the xy-plane with vertices (0,0), (4,1), (6,4) and (2,3).

Solution: A sketch of this region shows that it is a Type II region and would require three separate double integrals in either the *dy dx* or the *dx dy* ordering of integration. Instead, we observe that the region consists of two pairs of parallel sides, so we find equations for each of these sides:



- From (0,0) to (4,1), we have $y=\frac{1}{4}x$, or $-x+4y=0$.
- From (2,3) to (6,4), we have $y=\frac{1}{4}x+\frac{5}{2}$, or $-x+4y=10$.
- From (0,0) to (2,3), we have $y=\frac{3}{2}x$, or $-3x+2y=0$.
- From (4,1) to (6,4), we have $y=\frac{3}{2}x-5$, or $-3x+2y=-10$.



Thus, we can define a transformation from *x* and *y* into new variables *u* and *v* as

$$u=-x+4y \quad \text{so that} \quad 0\le u\le 10,$$
$$v=-3x+2y \quad \text{so that} \quad -10\le v\le 0.$$

The region of integration *R* transformed into the *uv*-plane is a rectangle, so that *u* and *v* have constant bounds.

We now rewrite this transformation so that *x* and *y* are isolated. We start with the system

$$u=-x+4y$$
$$v=-3x+2y.$$

Multiply the bottom row by -2:

$$\begin{aligned} u &= -x + 4y \\ -2v &= -2(-3x + 2y). \end{aligned}$$

Simplifying, we have

$$\begin{aligned} u &= -x + 4y \\ -2v &= 6x - 4y. \end{aligned}$$

Summing, the *y* terms sum to 0 and we have

$$u - 2v = 5x, \quad \text{so that} \quad x = \frac{1}{5}u - \frac{2}{5}v.$$

Substituting this back into the system, we have

$$y = \frac{3}{10}u - \frac{1}{10}v.$$

We can now find the Jacobian:

$$\begin{aligned} \mathrm{J}(u, v) &= \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \\ &= \begin{vmatrix} 1/5 & -2/5 \\ 3/10 & -1/10 \end{vmatrix} \\ &= \frac{1}{5}\left(-\frac{1}{10}\right) - \frac{3}{10}\left(-\frac{2}{5}\right) \\ &= -\frac{1}{50} + \frac{6}{50} \\ &= \frac{5}{50}, \qquad \text{or } \frac{1}{10}. \end{aligned}$$

We revisit the original double integral, and make substitutions:

$$\begin{aligned} \iint_R (x - 2y)\, dA &= \int_{-10}^{0}\int_{0}^{10} \left(\left(\frac{1}{5}u - \frac{2}{5}v\right) - 2\left(\frac{3}{10}u - \frac{1}{10}v\right)\right)\left(\frac{1}{10}\right) du\, dv\,. \\ &= \frac{1}{10}\int_{-10}^{0}\int_{0}^{10} \left(-\frac{2}{5}u - \frac{1}{5}v\right) du\ dv. \end{aligned}$$

The rest of the integration is routine calculation. The inside integral is evaluated first:

$$\int_0^{10} \left(-\frac{2}{5}u - \frac{1}{5}v\right)\, du = \left[-\frac{1}{5}u^2 - \frac{1}{5}uv\right]_0^{10} = -20 - 2v.$$

This is then integrated with respect to *v*. The Jacobian, $\frac{1}{10}$, has been moved to the front.

$$\frac{1}{10}\int_{-10}^{0} (-20 - 2v)\, dv = \frac{1}{10}[-20v - v^2]_{-10}^{0} = \frac{1}{10}(200 - 100) = 10.$$

This was an example of a *linear* transformation, in which the equations transforming *x* and *y* into *u* and *v* were linear, as were the equations reversing the transformation. In such a case, the Jacobian will be a constant.

We can also see how the geometry changed: The original region in the *xy*-plane has an area of 10 square units, while the region in the *uv*-plane, a rectangle where $0 \le u \le 10$ and $-10 \le v \le 0$, has an area of 100 square units. Thus, the region in the *uv*-plane is 10 times as large as the region in the *xy*-plane, so the Jacobian, $\frac{1}{10}$, “balances” this change in underlying area.

The Jacobian is usually taken to be a positive quantity. This is because the naming (and ordering) of the functions transforming *x* and *y* into *u* and *v*, then in reverse, is arbitrary. Since the Jacobian is a determinant, it is possible that two rows may be swapped depending on the original naming of the functions, which may introduce a factor of –1 into the result, which can be ignored.

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